MATH 430: NUMBER THEORY — SPRING 2021

EXAM 2 - TAKE HOME — 26 Mar 2021

- This is an open-book, open-note exam. You may use any proven result from our textbook with a proper **citation**: a reference to the page number and particular result being used. Using *unproven* statements (like unassigned exercises) is generally not allowed - some exam questions may have themselves been taken from the unassigned exercises.
- Likewise, if you use the internet to look up any definitions or theorems, you should keep a list of all webpages you visit and **cite** their web addresses/links with your submission.
- The instructor (me) reserves the right to ask any student to explain their answers to any or all questions on the exam. If the student is unable to provide a satisfactory answer, it will be assumed that the work submitted was not done in an earnest manner and the solution in question will receive no credit.

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**Question 1.** Define a new relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $x^2 + y^2$ is even.

- Prove that $\sim$ is an equivalence relation. Describe its equivalence classes.

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**Question 2.**

- Write **three** sentences that describe Carl Friedrich Gauss, when he lived, and which of his works first outlined the theory of modular arithmetic.
- Show that $2, 4, 6, \dots, 2m$ is a complete residue system modulo $m$ if $m$ is odd.
- Show that $1^2, 2^2, 3^2, \dots, m^2$ is a not complete residue system modulo $m$ for any $m > 2$.

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**Question 3.**

- Write **three** sentences describing Pierre de Fermat, when he lived, and his connection to Diophantus.
- Find the remainder of $23^{999}$ when divided by 13.
- Find the remainder of 24! when divided by 29.

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**Question 4.** Use the Fermat-Kraitchik method to factor the number $n = 426749$. (Hint: $653^2 = 426409$)

**Question 5.**

- Write **at least two** sentences that describe how the Chinese Remainder Theorem got its name.
- Find the least non-negative integer $x$ that simultaneously satisfies the following congruences:

$$\begin{aligned} 5x &\equiv 2 \mod 13 \\ x &\equiv 2 \mod 35 \\ 3x &\equiv 13 \mod 77 \\ x &\equiv 7 \mod 20 \end{aligned}$$

**Question 6.** Define a function to be **totally multiplicative** if it is multiplicative for *any* integers $m, n \in \mathbb{Z}$, without the extra condition that $m$ and $n$ be coprime.

- If a function $f(n)$ is totally multiplicative, is it necessarily true that

$$F(n) = \sum_{d|n} f(d)$$

  is also totally multiplicative? If so, provide a proof. If not, give a counterexample.

- If $\gcd(m, n) > 1$, prove that $\tau(mn) < \tau(m)\tau(n)$ and $\sigma(mn) < \sigma(m)\sigma(n)$.