Math 430: Number Theory — Spring 2021

Exam 1 - take home — 19 Feb 2021

- This is an open-book, open-note exam. You may use any proven result from our textbook with a proper **citation**: a reference to the page number and particular result being used. Using *unproven* statements (like unassigned exercises) is generally not allowed - some exam questions may have themselves been taken from the unassigned exercises.
- Likewise, if you use the internet to look up any definitions or theorems, you should keep a list of all webpages you visit and **cite** their web addresses/links with your submission.
- The instructor (me) reserves the right to ask any student to explain their answers to any or all questions on the exam. If the student is unable to provide a satisfactory answer, it will be assumed that the work submitted was not done in an earnest manner and the solution in question will receive no credit.

Question 1.

- Find **all** integer pairs (x, y) that satisfy the following linear Diophantine equation:

$$2340\,x + 18480\,y = 120.$$

- Find **any** integer triple (x, y, z) that satisfies the following linear Diophantine equation:

$$15x + 21y + 35z = 1.$$

- Write **three** sentences that describe who Diophantus was and why these equations are named for him.

Question 2.

- Let $\{n_1, n_2, \ldots, n_k\}$ be any finite set of integers. Define the terms **coprime** and **pairwise-coprime** in this context. Give an example of a set that illustrates the difference between the two definitions.
- Prove that one of the above conditions is "stronger" than the other, and explain what "stronger" means in this context.
- Suppose that a and b are positive integers whose sum $a + b$ is equal to a prime p.
  Prove that a and b are coprime.

Question 3. Prove that the following identity holds for all integers $n \geq 1$:

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n} = 3^n.$$

Question 4. If p is a prime and α is any nonnegative integer, introduce the new notation $p^\alpha \,\Big\|\, n$ to mean

$$p^\alpha \,\Big|\, n \quad \text{and} \quad p^{\alpha+1} \nmid n.$$

- Prove that if $p^\alpha \,\Big\|\, m$ and $p^\beta \,\Big\|\, n$, then $p^{\alpha+\beta} \,\Big\|\, mn$.
- Prove that if $p^\alpha \,\Big\|\, m$ and $p^\beta \,\Big\|\, n$ and $\alpha < \beta$, then $p^\alpha \,\Big\|\, m \pm n$.
- Find a counterexample to the assertion that, if $p^\alpha \,\Big\|\, m$ and $p^\alpha \,\Big\|\, n$, then $p^\alpha \,\Big\|\, m + n$.

Question 5. Prove that $\sqrt[3]{16 \cdot 27}$ is an irrational number.

Question 6.

- We know that there are infinitely many prime numbers. Write **three** sentences that describe where/when this fact was established and the person credited with its proof.
- Give another proof of the infinitude of primes by using the following integer to arrive at a contradiction:

$$N = (p_2 p_3 p_4 \cdots p_n) + (p_1 p_3 p_4 \cdots p_n) + (p_1 p_2 p_4 \cdots p_n) + \cdots + (p_1 p_2 p_3 \cdots p_{n-1}).$$