Midterm Practice Problems

This practice exam should take 90 minutes.

1. (Number of subsets. Show that a set of n elements has 2^n subsets.) Use proof by induction over $|A|$ to show the following claim: For any set A, $|\mathcal{P}(A)| = 2^{|A|}$.

2. Use proof by contradiction to show that $n^4 \notin \mathcal{O}(n^2)$. **You may not use the limit rule. Please use the formal definition of big O.**

3. Assume $x \in Z$ and $y \in \mathbb{N}$. If all y's can not be divided by x, then x is 0.

4. A Pythagorean triple consists of three positive integers a, b, and c such that $a^2 + b^2 = c^2$, commonly written as (a, b, c). When a Pythagorean triple a, b and c have a greatest common divisor of 1, the triple is called primitive. Given x, y, z is a primitive Pythagorean triple, show that the greatest common divisor between x and y is 1 and the greatest common divisor between x and z is 1 and the greatest common divisor between y and z is 1. (Note that the greatest common divisor of a and b is the largest positive integer that divides both a and b and must be greater than or equal to 1.)

5. In a town, the mayor announced that there are imposters among the citizens. Everyone knows each other's identity, but not their own. The town forbids people to share knowledge of each others' identity to ensure the imposters are not warned. During the day, the police will set up a hideout near one of the imposters' homes and will arrest all imposters at night when all hideouts are in place. Once the arrest is made, people will hear about the act the next morning. Show that given there are n imposters, an imposter will learn about their own identity on the nth morning.

6. We know the addition formula for the tangent is as follows,

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

   Prove or disprove $\tan 1^\circ$ is rational. You can use $\tan 30^\circ = \frac{1}{\sqrt{3}}$, $\tan 45^\circ = 1$ and $\tan 60^\circ = \sqrt{3}$ if you need.