Midterm

March 19, 2021

1. Let $a, b, c \in \mathbb{N}$. Prove the following using contraposition: if $a^2 + b^2 + c^2 \equiv 3 \mod 4$ then $a + b + c \not\equiv 0 \mod 4$.

   Proof: We prove the contrapositive. Assume that $a + b + c \equiv 0 \mod 4$, and show $a^2 + b^2 + c^2 \not\equiv 3 \mod 4$.

$$\begin{aligned} a+b+c &\equiv 0 \mod 4 \iff \\ (a+b+c)^2 &\equiv 0 \mod 4 \iff \\ a^2+b^2+c^2+2(ab+bc+ac) &\equiv 0 \mod 4 \iff \\ a^2+b^2+c^2+2k &\equiv 0 \mod 4 \text{ for } k = ab+bc+ac \end{aligned}$$

   As for any positive integer k we have that $2k$ is congruent to either 0 or 2 $\mod 4$, it means that $a^2 + b^2 + c^2$ will also be either congruent to 0 or 2 $\mod 4$. Therefore, $a^2 + b^2 + c^2 \not\equiv 3 \mod 4$, and the contrapositive is proved.

   5 pts: Writing out the contrapositive correctly
   2 pts: Reaching the statement $a^2 + b^2 + c^2 + 2(ab + bc + ac) \equiv 0 \mod 4$
   1.5 pts: Reaching the statement $2(ab + bc + ac) \equiv 0 \mod 4$
   1.5 pts: Reaching the statement $2(ab + bc + ac) \equiv 2 \mod 4$
   1.5 pts: Identifying $a^2 + b^2 + c^2 \in 0 \mod 4$
   1.5 pts: Identifying $a^2 + b^2 + c^2 \in 2 \mod 4$
   2pts: Inferring that the negation of the premise of the original statement follows, i.e. $a^2 + b^2 + c^2 \not\equiv 3 \mod 4$

2. Show that for any prime number $p > 2$ there is some prime number q strictly greater than p and smaller than $p!$. Prove this claim directly.

   Proof: We will prove the statement directly.

   Let all prime numbers smaller or equal than p be $p_1, p_2, \ldots, p_k \leq p$. Consider the number $r = p_1 p_2 \ldots p_k + 1$. r will not be divisible by any of $p_1, \ldots, p_k$, and thus will either be itself prime, or have a prime factor greater than p.

   Now, let us compare r with $p!$. We have

$$\begin{aligned} p! &\geq 1 * 2 * \cdots * (p_1 - 1) * p_1 * (p_1 + 1) \ldots (p_2 - 1) * p_2 * (p_1 + 2) \cdots * p_k \\ &> p_1 * p_2 \cdots * p_k \\ &= (r - 1) \end{aligned}$$

   So, we have $p! \geq r$. As we showed that there is a prime number greater than p and smaller or equal than r, it means this prime number will also be smaller than $p!$. This concludes our proof.

   2 pts: Identifying a candidate prime number r in the proof
   2 pts: ensuring/showing candidate prime $r < p!$
   2 pts: ensuring/showing candidate prime $r > p$
   3 pts: recognizing that if r is prime proof is complete
   2 pts: If r is not prime, identifying that there is another prime factor of r, f
   2 pts: If r is not prime, observing/demonstrating that $f < p!$
   2 pts: If r is not prime, observing/demonstrating that $f > p$

3. Use proof by contradiction to show that $n^2 \notin \Omega(n!)$. **You may not use the limit rule. Please use the formal definition of big Omega.**

   Assume $n^2 \in \Omega(n!)$. That means there exists $c > 0$ and $n_0 > 0$ such that $0 \leq cn! \leq n^2$ whenever $n \geq n_0$. Now, divide both sides of the equation by n and we have $0 \leq c(n-1)! \leq n$. Let $n^* = \lceil \max(n_0, \frac{1}{c}) \rceil + 4 \Rightarrow n^* \geq \frac{1}{c} + 4$. (The left inequality holds because $c > 0$ and $n_0 > 0$.) Let's consider the right inequality:

   $c(n^*-1)! \geq c\,(n^*-1)\,(n^*-2)\,(n^*-3)\,(n^*-4)!$ (by definition of factorial)
   $\geq (n^*-1)\,(n^*-2)\,(n^*-4)!$ (since $c \cdot (n^*-3) \geq 1$ because $n^* \geq \frac{1}{c} + 4$)
   $\geq (n^*-1)\,(n^*-2)$ (since $(n^*-4)! \geq 1$ because $n_0 > 0$ and $c > 0$)
   $\geq n^{*2} - 3n^* + 2$
   $\geq n^*(n^*-3) + 2$ (factoring)
   $\geq n^*(n^*-3) + 2 > n^*$ (since $n^* \geq 4$ implies $(n^*-3) \geq 1$)
   $\Rightarrow\Leftarrow$

   2pts: Write out the correct statement to prove by contradiction.
   3pts: Write out the correct formal definition of big Omega.
   5pts: Set a valid n^* value to break the inequality.
   - 1 pt attempt to set n^* even if incorrect
   - 2 pts attempt to set $n^* \geq 4$ even if incorrect
   - 2 pts for correct choice
   5pts: Correctly manipulate the inequality equation to show contradiction.
   -1 pt - any attempt to manipulate $n!$ even if incorrect.
   - 1 pt -correct inference to eliminate c
   - 1 pt -correct inference to upper bound factorial
   - 1 pt - correct expression for which to show contradication
   - 1 pt - correct inference to show expression greater than n
   0pts: Each minor algebraic mistake.
   0pts: No deduction for not using ceiling
   5 pts: Use of limit rules rather than the formal definition of big Omega (Solution must otherwise be correct.)

4. Use proof by induction over $|B|$ to show the following claim: For any sets A and B , $|A \times B| = |A| \times |B|$.

   B.C. For $|B| = 0$, $|A \times B| = |\emptyset| = 0 \times |A| = 0$.

   I.H. Assume for all $0 \leq n \leq k$ for some arbitrary fixed $k \geq 0$ that $|B| = n$ and $|A \times B| = |A| \times |B| = n|A|$.

   I.S. When $|B| = k+1$, since B is nonempty, $B = B' \cup \{a\}$ where a is an element of B and $B' = B - \{a\}$. The cardinality of the cartesian product can be written as,

$$\begin{aligned}|A \times B| &= |A \times (B' \cup \{a\})| \\ &= |(A \times B') \cup (A \times \{a\})| \\ &= |A \times B'| + |A \times \{a\}|\end{aligned}$$

   By I.H., since $|B'| = k$, $|A \times B'| = k|A|$. Let's consider $|A \times \{a\}|$. By the definition of the Cartesian product, for each element of A, x, this set will contain an ordered pair, $< x, a >$ with the element x first and the element a second. There will be $|A|$ such ordered pairs so $|A \times \{a\}| = |A|$ Accordingly,

$$|A \times B| = (k)|A| + |A| = |A|(k+1) = |A| \times |B|$$

   .

   2pts: Base case is correct (Need to start with 0)
   2pts: IH assumes $|B| = k$, $|A \times B| = |A| \times |B|$

1pts: IH indicates correct range (≥ 0).
3pts: partitioning of the set in the inductive step around an arbitrary element
2 pts: application of IH
2 pts:Use of definition of Cartesian product to explain why $|A \times \{a\}| = |A|$. This may be done as base case or in the inductive step
3 pts: Correct inference and algebraic manipulation showing in the inductive step that

$$|A \times B| = (k+1)|A|$$

-0pts: Each minor algebraic or set manipulation mistake.

5. Show that $4^{2n-1} + 3^{n+1}$ is divisible by 13 for all $n \geq 1$ by induction.

B.C. For $n = 1$, $4^{2n-1} + 3^{n+1}$ is 13, which is clearly divisible by 13.

I.H. For $n = k (k \geq 1)$, assume there exists an integer ℓ such that $4^{2k-1} + 3^{k+1} = 13\ell$.

I.S. For $n = k + 1$,

$$\begin{aligned} 4^{2k+1} + 3^{k+2} &= 4^{2k+1} + 3 \cdot 3^{k+1} \\ &= 16 \cdot 4^{2k-1} + 3 \cdot (3^{k+1}) \\ &= 13 \cdot 4^{2k-1} + 3 \cdot 4^{2k-1} + 3 \cdot (3^{k+1}) \\ &= 13 \cdot 4^{2k-1} + 3(4^{2k-1} + 3^{k+1}) \\ &= 13 \cdot 4^{2k-1} + 3(13\ell) \\ &= 13(4^{2k-1} + 3\ell) \end{aligned}$$

As $4^{2k-1} + 3\ell$ is an integer, $4^{2k+1} + 3^{k+2}$ is divisible by 13.

This concludes proof.

2pts: Base case is correct
2pts: IH assumes $4^{2k-1} + 3^{k+1}$ is divisible by 13.
2pts: IH indicates correct range (≥ 1).
2pts: starting IS from $4^{2k+1} + 3^{k+2}$
2pts: correct factoring to apply inductive hypothesis
2 pts: correct application of inductive hypothesis
3 pts: correct inference for divisibility by 13
0pts: Each minor algebraic mistake.

6. Prove the following claim by contradiction: If p,q and $\sqrt{2}p + \sqrt[3]{3}q$ are all rational number, show that $p = q = 0$. You can assume $\sqrt{2}$ and $\sqrt[3]{3}$ are both irrational.
Suppose

$$\sqrt{2}p + \sqrt[3]{3}q = r,$$

where r is a rational number.

$$\sqrt[3]{3}q = r - \sqrt{2}p$$

Taking cube of both side and simplifying them, we get

$$\sqrt{2}\left(3pr^2 + 2p^3\right) = r^3 + 6rp^2 - 3q^3.$$

If we assume $3pr^2 + 2p^3 \neq 0$, we can get

$$\sqrt{2} = \frac{r^3 + 6rp^2 - 3q^3}{3pr^2 + 2p^3}.$$

This contradicts $\sqrt{2}$ is irrational. So $3pr^2 + 2p^3 = 0$. As $3pr^2 + 2p^3 = p(3r^2 + 2p^2) = 0$, we knew $p = 0$ or $3r^2+2p^2 = 0$. That is equivalent to $p = 0$ or "$r = 0$ and $p = 0$". In either case, we can conclude $p = 0$.

Now substituting $p = 0$, we get

$$\sqrt[3]{3}q = r.$$

If we assume $q = 0$, we have $\sqrt[3]{3} = \frac{r}{q}$, which contradicts the assumption. So we get $q = 0$.

5 pts: Correct negation of the implication to begin the proof by contradiction.
3 pts: Using the representation of $\sqrt[3]{3}$ and $\sqrt{2}$ as rational numbers to infer a contradiction using irrationality. (i.e.$p \neq 0$ and $q \neq 0$ leads to contradiction.)
3 pts: Using the representation of $\sqrt[3]{3}$ and $\sqrt{2}$ as rational numbers to infer $p = 0$ and deriving contradiction using irrationality (i.e. $p = 0$ and $q \neq 0$ leads to contradiction)
3 pts: Using the representation of $\sqrt[3]{3}$ and $\sqrt{2}$ as rational numbers to infer/consider $q = 0$ and deriving contradiction using irrationality (i.e. $q = 0$ and $p \neq 0$ leads to contradiction)
1 pts: Using assumptions/infererence from previous cases such as $3pr^2 + 2p^3 \neq 0$ and $p = 0$ to derive contradiction that $q = 0$ (and $p = 0$).

7. A restaurant has a four course meal and for each course customers have a choice of three dishes per course and must choose exactly one dish. What is the fewest number of customers that the restaurant must have during a single evening dinner service to ensure that 4 customers order the exact same meal i.e. they ordered the same dish for each course for all four courses? Prove your answer.

   Let$A$ be the set of customers during dinner service and B the set of all possible meals during a single four course dinner service. $|B|$ is the size of the Cartesian product of the choices of dishes for each course, so $|B| = 3^4 = 81$.
   Let $f : A \to B$ be such that customers are mapped to their meal choices. We want to figure out the smallest $|A|$ such that at least one meal in B has (at least) 4 customers. By the extended PHP if A is such that $\lceil \frac{|A|}{|B|} \rceil = 4$, the problem will be satisfied. If $|A| = |B| \cdot 4$ the claim would be true, but it would not be the fewest customers and the claim would be false if $|A| = |B| \cdot 3$. In this case to ensure $\lceil \frac{|A|}{|B|} \rceil = 4$ let $|A| = |B| \cdot 3 + 1 = 244$ and claim follows.

   2 pts: specification of B (pigeonholes)
   3 pts: correct $|B|$
   1 pt: correct function mapping customers to meals
   3 pts correct application of extended PHP
   6 pts: correct value for $|A|$
   - 2 pts: mention why cannot be fewer
   - 2 pts: mention why must be less than number of possible meals times 4
   - 2 pts: correct use of ceiling function to get smallest cardinality of A (by adding one)

8. Let $f : X \to Y$ be a function and $f^{-1} : Y \to X$ be its inverse relation. f^{-1} is a bijective function. Show that f is a bijection.

   (a) Use proof by contradiction to show that f is injective.
   (b) Use proof by contradiction to show tht f is surjective.
   (c) You cannot simply cite any results shown in the textbook or class.
   (d) You must write your answer in as much quantificational logic as you can.You will not receive much credit for answers not written in quantificational logic.

(a) Use proof by contradiction to show that f is injective.
Assume $\exists x_1, x_2 \in X, \exists y \in Y(x_1 \neq x_2 \wedge f(x_1) = f(x_2) = y)$
$\implies < y, x_1 >\in f^{-1} \wedge < y, x_2 >\in f^{-1}$ (by definition of inverse relation)
$\implies x_1 = x_2$ (since f^{-1} is bijection (but function enough))
$\Rightarrow\!\Leftarrow$

(b) Use proof by contradiction to show that f is surjective.
Assume $\exists y \in Y, \forall x \in X < x, y >\notin f$
$\implies \exists y \in Y, \forall x \in X < y, x >\notin f^{-1}$ (by definition of inverse relation)
$\Rightarrow\!\Leftarrow$ (since f^{-1} is function)

Recall that f^{-1} being a function means that
$\forall y \in Y, \exists x \in X(< y, x >\in f^{-1} \wedge (\forall z \in X(x \neq z \implies < y, z >\notin f^{-1})))$

5 pts: quantificational logic (will give pt based on 20 percent increments)
2 pts: correct negation for showing f injective.
2 pts: correct inference towards showing contradiction
1 pt: valid contradicton
2 pts: correct negation for showing f surjective.
2 pts: correct inference towards showing contradiction
1 pt: valid contradicton