1. Let $X_1, \ldots, X_n$ be i.i.d., each with density function

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1,$$

where $\theta > 0$ is unknown. Find the following.

(a) The MLE of θ.

(b) The MLE's large-n asymptotic distribution.

(c) For given $\alpha \in (0,1)$, an approximate $(1-\alpha)$ confidence interval for θ using the MLE's asymptotic distribution.

Solution: The likelihood is

$$\theta^n (\prod X_i)^{\theta-1},$$

so the derivative of the log-likelihood is

$$\begin{aligned} \partial_\theta \log[\theta^n (\prod X_i)^{\theta-1}] &= \partial_\theta [n \log\theta + (\theta-1)\sum \log X_i] \\ &= \frac{n}{\theta} + \sum \log X_i, \end{aligned} \tag{1}$$

so the MLE satisfies

$$0 = \frac{n}{\theta} + \sum \log X_i \quad \Leftrightarrow \quad \widehat{\theta} = -\frac{n}{\sum \log X_i}.$$

Taking $n = 1$ *in* (1) *and another derivative, we have*

$$\partial_\theta^2 \log[\theta X_1^{\theta-1}] = -\frac{1}{\theta^2},$$

thus

$$I(\theta) = \frac{1}{\theta^2}.$$

By Theorem 8.5.2.B, the asymptotic distribution of $\widehat{\theta}$ *is*

$$N(\theta, \theta^2/n).$$

Finally, the approximate CI is

$$\widehat{\theta} \pm z_{\alpha/2} \frac{\widehat{\theta}}{\sqrt{n}}.$$

2. A scientist (who hasn't taken a statistics class in a while) gathers i.i.d. data $X_1, \ldots, X_n$ in his lab which follows the $N(\mu, 1)$ distribution.

(a) The scientist reports

$$\overline{X} \pm 1/\sqrt{n}$$

as a confidence interval for μ, where $\overline{X}$ is the sample mean. What is the actual confidence level of this interval? It may help to know that that standard normal c.d.f. takes the value $\Phi(1) = .84$.

(b) Suppose the scientist instead reports

$$\overline{X} \pm z_{.05}/\sqrt{n}$$

as a confidence interval for μ, where z_α denotes the upper α quantile of the standard normal distribution. What is the actual confidence level of this interval?

Solution:

(a) The $(1-\alpha)$*-CI is* $\overline{X} \pm z_{\alpha/2}/\sqrt{n}$ *so the effective value of* α *satisfies*

$$\begin{aligned} z_{\alpha/2} = 1 \Leftrightarrow 1 - \alpha/2 &= \Phi(1) = .84 \\ \Leftrightarrow 1 - \alpha &= 1 - 2(1 - .84) = 1 - 2(.16) = 1 - .32 = .68, \end{aligned}$$

so the confidence level is 68%*.*

(b) Here $z_{\alpha/2} = z_{.05}$ *so* $\alpha = 2(.05) = .1$*, so this is a* $1 - \alpha = 90\%$ *CI.*

3. Suppose that a single random variable X is observed, with density function

$$f(x|\theta) = (1-\theta)\theta^x, \quad x = 0, 1, 2, \ldots,$$

where $0 < \theta < 1$ is unknown.

(a) Find an expression for the c.d.f. of X. (*For this the geometric series*

$$\sum_{y=0}^{x} \theta^y = \frac{1 - \theta^{x+1}}{1 - \theta}$$

may be helpful, which you can use without proof.)

(b) Show that X is stochastically increasing in θ.

(c) Given $\alpha \in (0, 1)$, use the method of pivoting the c.d.f. to find a $(1 - \alpha)$ confidence interval for θ as a function of X.

(d) Write down the 62% confidence interval that results from observing $X = 1$. (*You should be able to get a numerical answer without a calculator.*)

Solution: For $x = 0, 1, 2, \ldots$*, using the hint we have*

$$P(X \le x) = \sum_{y=0}^{x} (1-\theta)\theta^y = (1-\theta)\frac{1-\theta^{x+1}}{1-\theta} = 1 - \theta^{x+1}.$$

This is decreasing in θ *since, for example, its partial derivative with respect to* θ *is*

$$-(x+1)\theta^x < 0.$$

Then to find the CI we try to solve,

$$\begin{aligned} \alpha/2 &= P_{\theta_U}(X \le x) = 1 - \theta_U^{x+1} \\ \alpha/2 &= P_{\theta_L}(X \ge x) = 1 - P_{\theta_L}(X \le x-1) = \theta_L^x \end{aligned} \tag{2}$$

or

$$\theta_L(x) = (\alpha/2)^{1/x} \quad \textit{and} \quad \theta_U(x) = (1-\alpha/2)^{1/(x+1)}. \tag{3}$$

For $x = 0$ *this last formula for* $\theta_U(x)$ *holds but we have to be a bit careful because the RHS of* (2) *equals* 1 *so we need to restrict to* $x \ge 1$ *there. For the* $x = 0$ *case, this is an instance where we need to take* $\theta_L(x)$ *to be the smallest possible value* θ *can take,* 0 *in this case. To summarize, the CI is* $(\theta_L(X), \theta_U(X))$ *where*

$$\theta_L(x) = \begin{cases} (\alpha/2)^{1/x}, & x \ge 1 \\ 0, & x = 0 \end{cases}$$

and $\theta_U(x)$ *is as in* (3)*. For* $\alpha = 1 - .62 = .38$ *we have*

$$(\theta_L(1), \theta_U(1)) = ((.38/2)^{1/1}, (1-.38/2)^{1/2}) = (.19, \sqrt{.81}) = (.19, .9).$$

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