1. Let $X_1, \ldots, X_n$ be i.i.d., each with density function

$$f(x|\theta) = \theta x^{\theta - 1}, \quad 0 < x < 1,$$

where $\theta > 0$ is unknown. Find the following.

(a) The MLE of $\theta$.

(b) The MLE's large-$n$ asymptotic distribution.

(c) For given $\alpha \in (0, 1)$, an approximate $(1 - \alpha)$ confidence interval for $\theta$ using the MLE's asymptotic distribution.

2. A scientist (who hasn't taken a statistics class in a while) gathers i.i.d. data $X_1, \ldots, X_n$ in his lab which follows the $N(\mu, 1)$ distribution.

(a) The scientist reports

$$\overline{X} \pm 1/\sqrt{n}$$

as a confidence interval for $\mu$, where $\overline{X}$ is the sample mean. What is the actual confidence level of this interval? It may help to know that that standard normal c.d.f. takes the value $\Phi(1) = .84$.

(b) Suppose the scientist instead reports

$$\overline{X} \pm z_{.05}/\sqrt{n}$$

as a confidence interval for $\mu$, where $z_\alpha$ denotes the upper $\alpha$ quantile of the standard normal distribution. What is the actual confidence level of this interval?

3. Suppose that a single random variable $X$ is observed, with density function

$$f(x|\theta) = (1 - \theta)\theta^x, \quad x = 0, 1, 2, \ldots,$$

where $0 < \theta < 1$ is unknown.

(a) Find an expression for the c.d.f. of $X$. (*For this the geometric series*

$$\sum_{y=0}^{x} \theta^y = \frac{1 - \theta^{x+1}}{1 - \theta}$$

*may be helpful, which you can use without proof.*)

(b) Show that $X$ is stochastically increasing in $\theta$.

(c) Given $\alpha \in (0, 1)$, use the method of pivoting the c.d.f. to find a $(1 - \alpha)$ confidence interval for $\theta$ as a function of $X$.

(d) Write down the 62% confidence interval that results from observing $X = 1$. (*You should be able to get a numerical answer without a calculator.*)

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