1. Let $X_1, \ldots, X_n$ be i.i.d. $N(0, \sigma^2)$ and $Y_1, \ldots, Y_m$ be i.i.d. $N(0, \tau^2)$ and independent of the X_i. For each of the following give the distribution (including the name and values of any parameters) as well as justification.

   (a)
$$\sum_{i=1}^{n} X_i$$

   (b)
$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

   (c)
$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

   (d)
$$\frac{\overline{X}}{S/\sqrt{n}}$$

   (e)
$$\sum_{i=1}^{m} Y_i^2$$

   (f)
$$\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{m} Y_i^2}$$

   Solution:

   (a) Being a linear combination of independent normals, this is normal with mean
$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} EX_i = \sum_{i=1}^{n} 0 = 0$$
   and variance
$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \sigma^2 = n\sigma^2.$$

   (b) By the same reasoning and similar calculations, $\overline{X} \sim N(0, \sigma^2/n)$.

   (c) By Theorem 6.3.B,
$$S^2 \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}.$$

(d) *Letting $Z \sim N(0,1)$ and $U \sim \chi^2_{n-1}$ be independent,*

$$\begin{aligned} \frac{\overline{X}}{S/\sqrt{n}} &= \frac{(\sigma/\sqrt{n})Z}{\sqrt{\sigma^2 U/(n(n-1))}} \quad \textit{[by parts (b) and (c)]} \\ &= \frac{Z}{\sqrt{U/(n-1)}} \\ &\sim t_{n-1}, \end{aligned}$$

this last by the definition of the t distribution. Alternatively, Corollary 6.3.B in the book can be used.

(e) *Writing $Y_i = \tau Z_i$ where the Z_i are independent standard normals,*

$$\sum_1^m Y_i^2 = \sum_1^m (\tau Z_i)^2 = \tau^2 \sum_1^m Z_i^2 \sim \tau^2 \cdot \chi^2_m,$$

this last by the definition of the χ^2 distribution.

(f) *By part (e), $\sum_1^n Z_i^2 \sim \sigma^2 \cdot \chi^2_n$, so letting U and V be independent χ^2 random variables with n and m d.f., respectively,*

$$\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^m Y_i^2} = \frac{\sigma^2 U}{\tau^2 V} = \frac{\sigma^2 n}{\tau^2 m} \cdot \frac{U/n}{V/m} \sim \frac{\sigma^2 n}{\tau^2 m} \cdot F_{n,m},$$

this last by the definition of the F distribution.

2. Let $X_1, \ldots, X_n$ be i.i.d. with the exponential distribution whose density function is

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1}$$

where $\theta > 0$ is unknown.

(a) Show that the moment generating function (MGF) of the distribution (1) is

$$M(t) = \frac{\theta}{\theta - t}. \tag{2}$$

(b) Name an open interval I containing 0 for which (2) is valid for all $t \in I$.

(c) Use the MGF to compute the first 2 moments of the exponential distribution (1).

(d) Find the Method of Moments estimator $\widehat{\theta}$ of θ.

(e) Use the approximation

$$\frac{1}{\overline{X}} \approx \theta - \theta^2(\overline{X} - 1/\theta) \tag{3}$$

to approximate the variance of $\widehat{\theta}$. Explain how you could use this to approximate the estimated standard error of $\widehat{\theta}$. You can use (3) without justification and assume it is accurate enough for these approximations.

Solution:

(a)

$$M(t) = E(e^{tX_1}) = \int_0^\infty e^{tx}\theta e^{-\theta x}dx = \theta\int_0^\infty e^{-(\theta-t)x}dx = \frac{\theta}{\theta - t}.$$

(b) We need $\theta - t > 0$ *so* $|t| < \theta$ *works.*

(c)

$$M'(t) = \frac{\theta}{(\theta-t)^2} \Rightarrow \mu_1 = M'(0) = 1/\theta,$$
$$M''(t) = \frac{2\theta}{(\theta-t)^3} \Rightarrow \mu_2 = M''(0) = 2/\theta^2.$$

(d) Since $\theta = 1/\mu_1$ *we have*

$$\widehat{\theta} = \frac{1}{\widehat{\mu}_1} = \frac{1}{\overline{X}}.$$

(e) By part (c),

$$Var(X_1) = \mu_2 - \mu_1^2 = 2/\theta^2 - 1/\theta^2 = 1/\theta^2.$$

Now, using the approximation,

$$Var(\widehat{\theta}) = Var(1/\overline{X}) \approx \theta^4\, Var(\overline{X}) = \theta^4\frac{Var(X_1)}{n} = \theta^4\frac{1/\theta^2}{n} = \frac{\theta^2}{n}.$$

Thus the standard error is $\theta/\sqrt{n}$ *and this can be estimated by*

$$\frac{\widehat{\theta}}{\sqrt{n}} = \frac{1}{\overline{X}\sqrt{n}}.$$