1. Let $X \sim \text{Bin}(n,p)$, the binomial distribution on $n \geq 2$ trials with probability $p$ of success. Recall that the density function of $X$ is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n.$$

   (a) Find the moment generating function (MGF) of $X$. It may help to recall the binomial theorem,

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

   for any numbers $a, b$.

   (b) Use the MGF to compute the first two moments of $X$.

*Solution: We have*

$$\begin{aligned} M(t) = Ee^{tX} &= \sum_x \binom{n}{x} (e^t p)^x (1-p)^{n-x} = (e^t p + 1 - p)^n, \\ M'(t) &= n(e^t p + 1 - p)^{n-1} e^t p, \\ M''(t) &= ne^t p[(e^t p + 1 - p)^{n-1} + (n-1)(e^t p + 1 - p)^{n-2} e^t p] \\ &= ne^t p(e^t p + 1 - p)^{n-2}[e^t p + 1 - p + (n-1)e^t p] \\ &= ne^t p(e^t p + 1 - p)^{n-2}[ne^t p + 1 - p]. \end{aligned}$$

*Then $EX = M'(0) = np$ and*

$$EX^2 = M''(0) = np(np + 1 - p) = np(1-p) + (np)^2.$$

2. For $0 \leq x \leq 1$, define the two c.d.f.s $F_0(x) = x^2$ and $F_1(x) = x^3$. Consider a random variable $X$ that has c.d.f. either $F_0$ or $F_1$, and suppose we want to test

$$H_0 : X \text{ has c.d.f. } F_0 \quad \text{vs.} \quad H_1 : X \text{ has c.d.f. } F_1.$$

   (a) Give the form of the rejection rule of the Neyman-Pearson test of these hypotheses as simply as possible, in terms of just $X$ and an undetermined critical value.

   (b) Given $\alpha \in (0,1)$, give the rejection rule of the level-$\alpha$ version of this test.

   (c) What is the power of the level-$\alpha$ test?

*Solution: The densities are $f_0(x) = 2x$ and $f_1(x) = 3x^2$, so*

$$\frac{f_0(X)}{f_1(X)} = \frac{2X}{3X^2} = \frac{2}{3X},$$

*and this is $\le C'$ iff $X \ge C$. For the level-$\alpha$ test, we need*

$$\alpha = P_0(X \ge C) = 1 - F_0(C) = 1 - C^2,$$

*so $C = \sqrt{1-\alpha}$. The power of this test is*

$$P_1(X \ge \sqrt{1-\alpha}) = 1 - F_1(\sqrt{1-\alpha}) = 1 - (1-\alpha)^{3/2}.$$

3. Suppose $Y$ is a nonnegative random variable with mean $EY = \mu$ and whose variance varies with its mean according to $\mathrm{Var}(Y) = \sigma^2(\mu) = \mu^3$. Find a variance stabilizing transformation $f$ such that the variance of the transformed variable $f(Y)$ is approximately constant in $\mu$.

   *Solution: We seek $f$ such that*

   $$f'(\mu) \propto \frac{1}{\sigma(\mu)} = \mu^{-3/2},$$

   *so $f(\mu) = 1/\sqrt{\mu}$ works and $f(Y) = 1/\sqrt{Y}$ is the transformed variable.*

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables (not necessarily normally distributed) with mean $\mu$ and variance $\sigma^2$, and let

   $$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

   be the sample mean. Use the delta method to find an approximation for

   $$\mathrm{Var}[(\overline{X}_n)^2]$$

   in terms of $\mu$, $\sigma$, and $n$.

   *Solution: We have $(\overline{X}_n)^2 = f(\overline{X}_n)$ where $f(x) = x^2$, and so $f'(x) = 2x$. We know that $E\overline{X}_n = \mu$, Var$(\overline{X}_n) = \sigma^2/n$, and the delta method tells us that*

   $$(\overline{X}_n)^2 = f(\overline{X}_n) \approx f(\mu) + f'(\mu)(\overline{X}_n - \mu)$$

   *so*

   $$Var[(\overline{X}_n)^2] \approx [f'(\mu)]^2\, Var(\overline{X}_n) = (2\mu)^2(\sigma^2/n) = 4\mu^2\sigma^2/n.$$

5. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ where both $\mu$ and $\sigma$ are unknown, and let $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$ denote the usual sample mean. We have discussed two different estimators for $\sigma^2$ in this situation, the sample variance

   $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2,$$

which is unbiased for $\sigma^2$, and the MLE of $\sigma^2$,

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Since both of these are proportional to $\sum_{i=1}^{n}(X_i - \overline{X})^2$, in this problem we will investigate the estimator

$$T_a = a\sum_{i=1}^{n}(X_i - \overline{X})^2,$$

where $a > 0$ is an arbitrary constant.

(a) What is the distribution of $T_a$? *Hint:* It may help to first recall the distribution of $S^2$.

(b) What is the bias of $T_a$ for estimating $\sigma^2$?

(c) What is the variance of $T_a$?

(d) Recall that the mean square error (MSE) of an estimator $\widehat{\theta}$ of $\theta$ is given by

$$\text{MSE}(\widehat{\theta}) = E(\widehat{\theta} - \theta)^2 = \text{Var}(\widehat{\theta}) + [\text{bias}(\widehat{\theta})]^2.$$

What is the MSE of $T_a$ for estimating $\sigma^2$?

(e) What value of $a$ minimizes the MSE of $T_a$?

*Solution:*

*(a) Since* $T_a = a(n-1)S^2$ *and* $S^2 \sim (\sigma^2/(n-1))\chi^2_{n-1}$ *by Theorem ??,*

$$T_a \sim a\sigma^2\chi^2_{n-1}.$$

*(b) Since the expectation of a* $\chi^2$ *is its d.f.,* $ET_a = a\sigma^2(n-1)$. *Thus,*

$$\textit{bias}(T_a) = ET_a - \sigma^2 = \sigma^2[a(n-1) - 1].$$

*(c) Since the variance of a* $\chi^2$ *is twice its d.f.,* $\textit{Var}(T_a) = a^2\sigma^4 \cdot 2(n-1)$.

*(d)*

$$\begin{aligned} \textit{MSE}(T_a) &= \textit{Var}(T_a) + [\textit{bias}(T_a)]^2 \\ &= 2a^2\sigma^4(n-1) + \sigma^4[a(n-1) - 1]^2 \\ &= \sigma^4[a^2(n-1)(2 + n - 1) - 2a(n-1) + 1] \\ &= \sigma^4[a^2(n-1)(n+1) - 2a(n-1) + 1]. \end{aligned} \tag{1}$$

*(e) Dropping the* $\sigma^4$ *in this last expression, differentiating w.r.t.* $a$ *and setting equal to 0 yields*

$$0 = 2a(n-1)(n+1) - 2(n-1) = 2(n-1)[a(n+1) - 1],$$

*so* $a = 1/(n+1)$ *is the minimizer, noting that* (1) *is quadratic in* $a$.