1. Let $X \sim \text{Bin}(n, p)$, the binomial distribution on $n \geq 2$ trials with probability p of success. Recall that the density function of X is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n.$$

   (a) Find the moment generating function (MGF) of X. It may help to recall the binomial theorem,

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

   for any numbers a, b.

   (b) Use the MGF to compute the first two moments of X.

2. For $0 \leq x \leq 1$, define the two c.d.f.s $F_0(x) = x^2$ and $F_1(x) = x^3$. Consider a random variable X that has c.d.f. either F_0 or F_1, and suppose we want to test

$$H_0 : X \text{ has c.d.f. } F_0 \quad \text{vs.} \quad H_1 : X \text{ has c.d.f. } F_1.$$

   (a) Give the form of the rejection rule of the Neyman-Pearson test of these hypotheses as simply as possible, in terms of just X and an undetermined critical value.

   (b) Given $\alpha \in (0, 1)$, give the rejection rule of the level-α version of this test.

   (c) What is the power of the level-α test?

3. Suppose Y is a nonnegative random variable with mean $EY = \mu$ and whose variance varies with its mean according to $\text{Var}(Y) = \sigma^2(\mu) = \mu^3$. Find a variance stabilizing transformation f such that the variance of the transformed variable $f(Y)$ is approximately constant in μ.

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables (not necessarily normally distributed) with mean μ and variance σ^2, and let

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

   be the sample mean. Use the delta method to find an approximation for

$$\text{Var}[(\overline{X}_n)^2]$$

   in terms of μ, σ, and n.

5. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ where both μ and σ are unknown, and let $\overline{X} = (1/n)\sum_{i=1}^n X_i$ denote the usual sample mean. We have discussed two different estimators for σ^2 in this situation, the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2,$$

which is unbiased for σ^2, and the MLE of σ^2,

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2.$$

Since both of these are proportional to $\sum_{i=1}^n (X_i - \overline{X})^2$, in this problem we will investigate the estimator

$$T_a = a\sum_{i=1}^n (X_i - \overline{X})^2,$$

where $a > 0$ is an arbitrary constant.

(a) What is the distribution of T_a? *Hint:* It may help to first recall the distribution of S^2.

(b) What is the bias of T_a for estimating σ^2?

(c) What is the variance of T_a?

(d) Recall that the mean square error (MSE) of an estimator $\widehat{\theta}$ of θ is given by

$$\mathrm{MSE}(\widehat{\theta}) = E(\widehat{\theta} - \theta)^2 = \mathrm{Var}(\widehat{\theta}) + [\mathrm{bias}(\widehat{\theta})]^2.$$

What is the MSE of T_a for estimating σ^2?

(e) What value of a minimizes the MSE of T_a?