Practice Midterm 1 solutions

You have 50 minutes. You may use one, one-sided sheet of notes. You may not use any calculator, cell phone, or similar device.

Name: **Date:**

Problem	Score
#1	/10
#2	/10
#3	/10
#4	/10
#5	/10
Total	/50

Problem 1: (a) Find an equation for the plane passing through the points $P = (3, 2, 2)$, $Q = (5, -1, 1)$, and $R = (-1, 0, -4)$.

This plane contains the following vectors:

$PQ = \langle 5-3, -1-2, 1-2\rangle = \langle 2, -3, -1\rangle$, $\qquad PR = \langle -1-3, 0-2, -4, -2\rangle =\rangle -4, -2, -6\rangle$.

Therefore the plane is normal to the following vector:

$$\mathbf{n} = PQ \times PR = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ -4 & -2 & -6 \end{vmatrix} = \langle\rangle = \langle 16, 16, -16\rangle.$$

It follows that we can express our plane by the following equation, which we simplify:

$$\begin{aligned} \mathbf{n} \cdot (\langle x, y, z\rangle - P) = 0 \\ \iff 16(x-3) + 16(y-2) - 16(z-2) = 0 \\ \iff x + y - z = 3. \end{aligned}$$

(b) Define C to be the curve C parametrized by $\mathbf{r}(t) = \langle 1 - t^2, t+1, 2t^2 + t + 2\rangle$. Find an equation for the line which is tangent to C at $(0, 2, 5)$.

First, we need to find the t for which $\mathbf{r}(t) = \langle 0, 2, 5\rangle$. Looking at the second coordinate, we have $t + 1 = 2$, hence $t = 1$. And indeed, $\mathbf{r}(1) = \langle 0, 2, 5\rangle$.

Next, to find a vector parallel to the tangent line, we compute $\mathbf{r}'(1)$:

$$\begin{aligned} \mathbf{r}'(t) = \langle -2t, 1, 4t+1\rangle \\ \implies \mathbf{r}'(1) = \langle -2, 1, 5\rangle. \end{aligned}$$

We now know a point on the tangent line and also a vector parallel to it, so we can write a vector equation for it:

$$\begin{aligned} L(t) = \mathbf{r}(1) + t\mathbf{r}'(1) \\ \implies L(t) = \langle 0, 2, 5\rangle + t \cdot \langle -2, 1, 5\rangle. \end{aligned}$$

We can also express it in terms of parametric equations:

$$\begin{gathered} x = -2t, \quad y = 2 + t \\ z = 5 + 5t. \end{gathered}$$

(c) Find the point where the line in (b) intersects the plane in (a).

We substitute the parametric equations we found in (b) into the equation for the plane that we found in (a):

$$\begin{aligned} (-2t) + (2+t) - (5+5t) = 3 \\ \iff -6t - 3 = 3 \\ \iff t = -1. \end{aligned}$$

The point of intersection is therefore $L(-1) = \langle -2, 1, 0\rangle$.

Problem 2: Consider the curve

$$\mathbf{r}(t) = \left\langle e^t, \frac{\sqrt{2}}{2}e^{2t}, \frac{1}{3}e^{3t} \right\rangle.$$

(a) Compute the length of $\mathbf{r}(t)$, for $0 \le t \le 3$.

First, we compute $|\mathbf{r}'(t)|$:

$$\mathbf{r}'(t) = \langle e^t, \sqrt{2}e^{2t}, e^{3t}\rangle = e^t\langle 1, \sqrt{2}e^t, e^{2t}\rangle$$
$$\implies |\mathbf{r}'(t)| = e^t\sqrt{1 + 2e^{2t} + e^{4t}} = e^t\sqrt{(1+e^{2t})^2} = e^{3t} + e^t.$$

We use this to compute the length of $\mathbf{r}(t)$, for $0 \le t \le 3$:

$$L = \int_0^3 |\mathbf{r}'(t)|\,\mathrm{d}t = \int_0^3 \left(e^{3t} + e^t\right)\mathrm{d}t = \left[\frac{1}{3}e^{3t} + e^t\right]_0^3 = \frac{1}{3}e^9 + e^3 - \frac{4}{3}.$$

(b) Suppose that $\rho(s)$ is the reparametrization by arclength of the curve $\mathbf{r}(t)$. Find the length of $\rho(s)$ for $0 \le s \le 7$. (Hint: you should not need to do any complicated calculations.)

By the definition of the reparametrization by arclength, the length of $\rho(s)$ for $0 \le s \le 7$ is $7 - 0 = 7$. With more detail, $|\rho'(s)| = 1$, so the length formula gives $\int_0^7 1\,\mathrm{d}e = 7$.

Problem 3: (a) Define f by

$$f(x,y) = \arctan\left(\log\left(\sqrt{x} + \frac{\cos x}{x^x}\right) - \pi^{1/x}\right) - x^2 y.$$

Compute the partial derivative f_{xxy}. *(Hint: There is a reason that you are only given 1.5".)*

We use Clairaut's theorem:

$$f_{xxy} = f_{yxx} = (-x^2)_{xx} = (-2x)_x = -2.$$

(b) Suppose $u = x^2y^3 + z^4$, where $x = p + 3p^2$, $y = pe^p$, and $z = p\sin p$. Use the chain rule to find u_p.

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}p} &= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}p} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}p} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}p} \\
&= (2xy^3)(1+6p) + (3x^2y^2)(e^p + pe^p) + (4z^3)(\sin p + p\cos p) \\
&= 2(p+3p^2)(pe^p)^3(6p+1) + 3p^2(p+3p^2)^2e^{3p}(p+1) + 4(p\sin p)^3(p\cos p + \sin p).
\end{aligned}$$

Problem 4: Let S be the surface in $\mathbb{R}^3$ defined by the equation

$$xz^2 - \arctan(yz) = -\frac{\pi}{4}.$$

(a) Find expressions for $\partial z/\partial x$ and $\partial z/\partial y$. (Recall that $(\arctan u)' = 1/(1+u^2)$.)

As suggested by the statement of (a), we think of $z = z(x,y)$ as a function of x and y. We cannot solve the defining equation for z, so we must implicitly differentiate. Set $F(x,y,z) := xz^2 - \arctan(yz)$. We first compute the partial derivatives of F:

$$F_x = z^2, \quad F_y = -\frac{z}{1+y^2z^2}, \quad F_z = 2xz - \frac{y}{1+y^2z^2}.$$

We can now compute $\partial z/\partial x$ and $\partial z/\partial y$:

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = \frac{-z^2}{2xz - \frac{y}{1+y^2z^2}} = \frac{-z^2(1+y^2z^2)}{2xz(1+y^2z^2) - y}, \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = \frac{\frac{z}{1+y^2z^2}}{2xz - \frac{y}{1+y^2z^2}} = \frac{z}{2xz(1+y^2z^2) - y}. \end{aligned}$$

(b) Determine whether $(0,1,1)$ lies on S. Using linear approximation, find an approximation of the z-coordinate of the point on S that has $x = -0.1$ and $y = 1.1$.

Plugging $(x,y,z) = (0,1,1)$ into the left-hand side of the defining equation of S, we have $0 \cdot 1^2 - \arctan(1 \cdot 1) = -\pi/4$, so $(0,1,1)$ does indeed lie on S. Continuing to view $z = z(x,y)$ as a function of x and y, we have the following linear approximation of z near $(0,1,1)$:

$$z \approx 1 + \frac{\partial z}{\partial x}(0,1) \cdot (x-0) + \frac{\partial z}{\partial y}(0,1) \cdot (y-1).$$

We have $\frac{\partial z}{\partial x}(0,1) = 2$ and $\frac{\partial z}{\partial y}(0,1) = -1$ (by substituting $x = 0, y = 1$ into our answer to (a)), so we obtain:

$$z \approx 1 + 2 \cdot (-0.1 - 0) - (1.1 - 1) = 0.7.$$

(c) Consider a path $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ lying on S that has $\mathbf{r}(0) = \langle 0,1,1 \rangle$. Assume that $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = -2$ and $\frac{\mathrm{d}y}{\mathrm{d}t}(0) = 1$. Find the value of $\frac{\mathrm{d}z}{\mathrm{d}t}(0)$.

Using the chain rule, we have:

$$\frac{\partial z}{\partial t}(0) = \frac{\partial z}{\partial x}(0,1,1) \cdot \frac{\mathrm{d}x}{\mathrm{d}t}(0) + \frac{\partial z}{\partial y}(0,1,1) \cdot \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 2 \cdot (-2) + (-1) \cdot 1 = -5.$$

Problem 5: Let S be the hyperboloid defined by $x^2 + y^2 - z^2 = 1$.

(a) Find an equation of the tangent plane P to S at $(1, 1, 1)$.

Set $F(x, y, z) \coloneqq x^2 + y^2 - z^2$. Then $\nabla F = \langle 2x, 2y, -2z\rangle$, and $\nabla F(1, 1, 1) = \langle 2, 2, -2\rangle$ is normal to P. We know that the point $(1, 1, 1)$ lies on P, so we have the following equation for P, which we simplify:

$$\begin{aligned}\mathbf{n} \cdot (\langle x, y, z\rangle - P) &= 0 \\ \iff 2(x - 1) + 2(y - 1) - 2(z - 1) &= 0 \\ \iff x + y - z &= 1.\end{aligned}$$

(b) Find all points p on S such that the tangent plane to S at p is parallel to P.

At a point $p = (x, y, z)$ on S, the tangent plane is normal to $\nabla F(p) = \langle 2x, 2y, -2z\rangle$. We must therefore find all (x, y, z) on S such that $\langle x, y, -z\rangle$ is parallel to $\langle 1, 1, -1\rangle$. (We have divided both vectors by 2, since that scalar factor does not affect whether they are parallel.) We compute the cross-product of these vectors:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & -z \\ 1 & 1 & -1 \end{vmatrix} = \langle -y + z, x - z, x - y\rangle.$$

This vector equals $\langle 0, 0, 0\rangle$ exactly when $x = y = z$. We must therefore find all points on S with $x = y = z$. Substituting $y = x, z = x$ into the defining equation of S, we obtain $x^2 = 1$, hence $x = \pm 1$. It follows that the two points on S where the tangent plane is parallel to P are $(1, 1, 1)$ and $(-1, -1, -1)$.

Here is an alternate way to find all points where $\langle x, y, -z\rangle$ and $\langle 1, 1, -1\rangle$ are parallel: two vectors are parallel if and only if one can be written as a scalar multiple of the other, so we must have $\langle x, y, -z\rangle = \lambda\langle 1, 1, -1\rangle$, hence $x = \lambda, y = \lambda, z = \lambda$. This is possible exactly when $x = y = z$.