Practice Midterm 2 for MATH 226, section 39559

You have 50 minutes.

Name: **Date:**

Problem	Score
#1	/10
#2	/10
#3	/10
#4	/10
#5	/10
Total	/50

Problem 1 (10 points): Define

$$f(x,y,z) = \frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2},$$

where $x > 0$, $y > 0$, and $z > 0$. Use Lagrange multipliers to find the minimum value of f on the portion of the surface $x^2 + y^2 + z^2 = 36$ with $x > 0, y > 0, z > 0$. (You may assume that this absolute minimum exists, and that the solution to the system of equations coming from Lagrange multipliers is this absolute minimum.)

Let's use Lagrange multipliers with $g(x,y,z) = x^2 + y^2 + z^2$ to find the critical points of f subject to the constraint $g = 36$:

$$\begin{cases} -2/x^3 & = 2\lambda x, \\ -8/y^3 & = 2\lambda y, \\ -18/z^3 & = 2\lambda z, \\ x^2 + y^2 + z^2 & = 36. \end{cases}$$

Solving for λ (and using that x, y, z are nonzero), we have $-1/x^4 = -4/y^4 = -9/z^4$, hence $y = \sqrt{2}x$, $z = \sqrt{3}x$ (here we have used that x, y, z are all positive). Substituting this into the constraint, we have $x^2 + 2x^2 + 3x^2 = 36$, hence $x = \sqrt{6}$, hence the one critical point of f subject to $g = 36$ is $\left(\sqrt{6}, 2\sqrt{3}, 3\sqrt{2}\right)$. This must be the global minimum, since as we approach the boundary of this portion of the ellipsoid, f goes to ∞. Therefore the global minimum of f on this eighth-ellipsoid is $f\left(\sqrt{6}, 2\sqrt{3}, 3\sqrt{2}\right) = 1/6 + 4/12 + 9/18 = 1$.

Problem 2 (10 points): Consider the function

$$f(x,y) = \frac{1}{5}xy^2 - x$$

on the domain $D = \{(x,y) \mid -2 \le x \le 2,\ x^2 \le y \le 4\}$.

(a; 3 points) Find the critical points of $f(x,y)$ in the interior of D.

The critical points are where f_x and f_y both vanish. We compute these partial derivatives:

$$f_x = \frac{1}{5}y^2 - 1, \qquad f_y = \frac{2}{5}xy.$$

The equation $f_x = 0$ gives $y = \pm\sqrt{5}$, hence the equation $f_y = 0$ gives $x = 0$. Of the points $(0, \pm\sqrt{5})$, only the point $(0, \sqrt{5})$ is in D.

(b; 3 points) Classify the critical points of $f(x,y)$ in the interior of D as local minima, local maxima, or saddles.

We use the second derivative test. We compute the quantity called "D" (unfortunate overlap of notation):

$$\begin{aligned} D = f_{xx}f_{yy} - f_{xy}^2 = \left(\frac{1}{5}y^2 - 1\right)_x \left(\frac{2}{5}xy\right)_y - \left(\left(\frac{1}{5}y^2 - 1\right)_y\right)^2 &= 0 \cdot \left(\frac{2}{5}x\right) - \left(\frac{2}{5}y\right)^2 \\ &= -\frac{4}{25}y^2. \end{aligned}$$

It follows that $D(0, \sqrt{5}) = -4/5$ is negative, hence $(0, \sqrt{5})$ is a saddle point.

Problem 2 (continued).

(c; 4 points) Find the absolute maximum and minimum values of $f(x, y)$ on D.

We must find the candidate-extrema on the interior and boundary. The only candidate on the interior is the critical point $(0, \sqrt{5})$, but we know that it cannot be a global extremum of f on D because $(0, \sqrt{5})$ is a saddle.

The boundary is composed of two curves: L_1, the curve (x, x^2) for $-2 \leq x \leq 2$, and L_2, the curve $(x, 4)$ for $-2 \leq x \leq 2$.

- Consider L_1. We have $f(x, x^2) = \frac{1}{5}x^5 - x$. Its critical points are where $(\frac{1}{5}x^5 - x)' = (x^4 - 1) = 0$, i.e. ± 1. Its endpoints are $(\pm 2, 4)$.
- Consider L_2. We have $f(x, 4) = \frac{16}{5}x - x = \frac{11}{5}x$, hence it has no critical points. Its endpoints are $(\pm 2, 4)$.

The candidate-extrema are therefore $(1, 1)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$ (we argued that the interior critical point cannot be a global extremum, because it is a saddle). We have $f(1, 1) = -\frac{4}{5}$, $f(-1, 1) = \frac{4}{5}$, $f(2, 4) = \frac{22}{5}$, and $f(-2, 4) = -\frac{22}{5}$, so the absolute max res. min are $\frac{22}{5}$ and $-\frac{22}{5}$.

Problem 3 (10 points): Evaluate the following integrals.

(a; 5 points) $I = \int_0^4 \int_{\sqrt{x}}^2 \frac{3}{y^3+1}\, dy\, dx$ (switch the order!).

By drawing the domain of integration, we see that we can swap the order and rewrite the integral like so:

$$I = \int_0^4 \int_{\sqrt{x}}^2 \frac{3}{y^3+1}\, dy\, dx = \int_0^2 \int 0^{y^2} \frac{3}{y^3+1}\, dx\, dy = \int_0^2 \frac{3y^2}{y^3+1}\, dy = \int_1^9 \frac{1}{u}\, du = \log 9.$$

(b; 5 points) $I = \int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \sqrt{1+x^2+y^2}\, dy\, dx.$

The upper bound on y is $y = \sqrt{4-x^2}$, which yields $x^2+y^2 = 4$, i.e. the circle of radius 2. The lower bound is $y = x$, i.e. the line that makes an angle of $\pi/4$ with the $+x$-axis. In polar coordinates, the domain of integration is therefore

$$E = \{(r,\theta) \mid \pi/4 \le \theta \le \pi/2, 0 \le r \le r\}\,.$$

We compute I in polar coordinates:

$$\begin{aligned} I = \int_{\pi/4}^{\pi/2} \int_0^2 r\sqrt{1+r^2}\, dr\, d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{2} \int_1^5 \sqrt{u}\, du\, d\theta &= \int_{\pi/4}^{\pi/2} \frac{1}{2} \left[\frac{2}{3}u^{3/2}\right]_1^5 d\theta \\ &= \int_{\pi/4}^{\pi/2} \frac{1}{3}\left(5^{3/2}-1\right) d\theta \\ &= \frac{\pi}{12}\left(5^{3/2}-1\right). \end{aligned}$$

Problem 4 (10 points): Consider the region E that is under the plane $z + 3y = 16$, inside the cylinder $x^2 + y^2 = 4$, and in the first octant.

(a; 3 points) Rewrite the integral $\iiint_E x\,dV$ as an iterated integral in Cartesian coordinates (i.e. x, y, z). (Do not compute this integral.)

The constraint that E is under the plane $z + 3y = 16$ is equivalent to $z \leq -3y + 16$. The constraint that E is inside the given cylinder is equivalent to $x^2 + y^2 + \leq 4$. Finally, the constrain that E is in the first octant is equivalent to $x \geq 0$, $y \geq 0$, and $z \geq 0$. We can therefore rewrite E like so:

$$E = \left\{0 \leq x \leq 2,\ 0 \leq y \leq \sqrt{4 - x^2},\ 0 \leq z \leq -3y + 16\right\}.$$

This allows us to rewrite the integral I like so:

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{-3y+16} x\,dz\,dy\,dx.$$

(b; 3 points) Rewrite this integral as an iterated inteigral in cylindrical coordinates.

This is pretty straightforward, as long as remember the extra factor of r we pick up when going into cylindrical coordinates:

$$I = \int_0^{\pi/2} \int_0^2 \int_0^{-3r\sin\theta+16} r^2 \cos\theta\,dz\,dr\,d\theta.$$

(c; 4 points) Evaluate this integral.
Aye-aye:

$$\begin{aligned}
I = \int_0^{\pi/2} \int_0^2 \int_0^{-3r\sin\theta+16} r^2 \cos\theta\,dz\,dr\,d\theta &= \int_0^{\pi/2} \int_0^2 \left(-3r^3 \sin\theta\cos\theta + 16r^2\cos\theta\right) dr\,d\theta \\
&= \int_0^{\pi/2} \left[-\tfrac{3}{4}r^4 \sin\theta\cos\theta + \tfrac{16}{3}r^3\cos\theta\right]_{r=0}^{2} d\theta \\
&= \int_0^{\pi/2} \left(-12\sin\theta\cos\theta + \tfrac{128}{3}\cos\theta\right)\,d\theta \\
&= \left[-6\sin^2\theta + \tfrac{128}{3}\sin\theta\right]_{\theta=0}^{\theta=\pi/2} \\
&= -6 + \tfrac{128}{3} \\
&= \tfrac{110}{3}.
\end{aligned}$$

Problem 5 (10 points):

(a; 5 points) Rewrite the integral $I = \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$ as an iterated integral in the order $dx\,dy\,dz$. (You may want to draw a picture of the domain of integration, to help you figure out how to switch the order.)

This is tantamount to rewriting the domain:

$$\begin{aligned} E &= \big\{(x,y,z) \mid -1 \le x \le 1,\ x^2 \le y \le 1,\ 0 \le z \le 1-y\big\} \\ &= \big\{(x,y,z) \mid 0 \le z \le 1,\ 0 \le y \le 1-z,\ -\sqrt{y} \le x \le y\big\}. \end{aligned}$$

We can therefore rewrite I in the desired order:

$$I = \int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

(b; 5 points) Compute the integral $I = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{1} z\,dz\,dx\,dy.$

We rewrite the domain in cylindrical coordinates:

$$\begin{aligned} E &= \left\{(x,y,z) \mid -1 \le y \le 1,\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2},\ \sqrt{x^2+y^2} \le z \le 1\right\} \\ &= \{(r,\theta,z) \mid 0 \le \theta \le 2,\ 0 \le r \le 1,\ r \le z \le 1\}\,. \end{aligned}$$

We now compute I:

$$\begin{aligned} I = \int_0^{2\pi} \int_0^1 \int_r^1 rz\,dz\,dr\,d\theta &= \int_0^{2\pi} \int_0^1 \frac{1}{2} r(1-r^2)\,dr\,d\theta \\ &= \int_0^{2\pi} \left[\left(\frac{1}{4}r^2 - \frac{1}{8}r^4\right)\cos\theta\right]_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{8}\,d\theta = \frac{\pi}{4}. \end{aligned}$$