Midterm 2 for MATH 226, section 39559

You have 50 minutes.

You may use any resources (textbook, internet, notes, etc.) except that you may not consult any other human.

Show your work! Correct answers with no work may not get any credit.

You may use a calculator to aid your computation, but your final answer should be exact, e.g., π as opposed to 3.14.

Name: **Date:**

Problem	Score
#1	/10
#2	/10
#3	/10
#4	/10
#5	/10
Total	/50

(There is an opportunity for up to 2 points of extra credit on problem 5!)

Problem 1: Consider the function $f(x,y) \coloneqq x^4 + xy + y^4$, with domain the *open* region $D \coloneqq \left\{(x,y)\colon y < x + \frac{1}{2}\right\}$.

(a; 5 points) Find the critical points of f on D.

Let (X,Y) be a critical point. Then

$$\nabla f(X,Y) = \left\langle 4X^3 + Y, 4Y^3 + X \right\rangle = \langle 0,0 \rangle,$$

which is if and only if

$$4X^3 + Y = 0 \iff Y = -4X^3,$$

and

$$4Y^3 + X = 0 \iff X = -4Y^3.$$

simultaneously. Substituting $-4Y^3$ in X, we obtain

$$Y = -4(-4Y^3)^3 = 4^4Y^9 = 2^8Y^9 \iff [(2Y)^8 - 1]Y = 0.$$

Hence either $Y = 0$ or $(2Y)^8 = 1$, or $Y = \pm\frac{1}{2}$. When $Y = 0$, the corresponding $X = 0$, and for $Y = 1/2$ and $Y = -1/2$, $X = -1/2$ and $X = 1/2$, respectively. Hence the critical points in all of $\mathbb{R}^2$ are: $(0,0)$, $(\frac{1}{2}, -\frac{1}{2})$, and $(-\frac{1}{2}, \frac{1}{2})$. Note that only $(0,0)$ and $(\frac{1}{2}, -\frac{1}{2})$ are in D.

(b; 5 points) Classify the critical points you found in (a) as local maxima, local minima, or saddles.

To classify these critical points, we use the second derivative test. Note that

$$D(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 144x^2y^2 - 1.$$

At $(0,0)$, we have $D(0,0) = -1 < 0$, so the critical point $(0,0)$ is a saddle. At $(\frac{1}{2}, -\frac{1}{2})$, we have

$$D\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{144}{16} - 1 = 8 > 0,$$

and $f_{xx}(\frac{1}{2}, -\frac{1}{2}) = 12(\frac{1}{2})^2 = 3 > 0$. This implies that the point $(\frac{1}{2}, -\frac{1}{2})$ is a point of local minimum.

Problem 2: Consider the tasks of finding the point on the sphere $x^2 + y^2 + z^2 = 1$ that are at the greatest distance (respectively the least distance) from the point $(0, 1, 2)$.

(a; 5 points) Write down a system of equations, whose solutions are the candidates for the points on this sphere that are the farthest from (respectively, closest to) $(0, 1, 2)$. *(Hint: use Lagrange multipliers, applied to the squared distance function.)*

We would like to maximize/minimize the distance which is non-negative, therefore we might as well maximize/minimize the squared distance function. Note that the squared distance $D(x, y, z)$ between a generic point (x, y, z) and $(0, 1, 2)$ is

$$D(x, y, z) = (x - 0)^2 + (y - 1)^2 + (z - 2)^2.$$

Observe that

$$\nabla D(x, y, z) = \langle 2x, 2(y - 1), 2(z - 2) \rangle .$$

Since the points (x, y, z) must be on the sphere $x^2 + y^2 + z^2 = 1$, it translates to the constraint $g(x, y, z) = 1$ where $g(x, y, z) = x^2 + y^2 + z^2$. Hence the system we need to solve is $\nabla D = \lambda \nabla g$ together with $g = 1$, which is

$$\begin{aligned} 2x &= 2\lambda x; \\ 2(y - 1) &= 2\lambda y; \\ 2(z - 2) &= 2\lambda z; \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

(b; 5 points) Find these points, and identify which is the farthest from $(0, 1, 2)$ and which is the closest.

Observe that when $x \neq 0$, the first equation gives $\lambda = 1$. For this λ, the second equation becomes $-1 = 0$, which is impossible. Hence $x = 0$. This gives $y^2 + z^2 = 1$.

Massaging the second and the third equations give the equations $y(\lambda - 1) = 1$ and $z(\lambda - 1) = 2$. Since none of the factors $y, z, \lambda - 1$ can be zero, we may divide the first equation by second, yielding

$$\frac{y}{z} = \frac{1}{2} \iff z = 2y.$$

Substituting this result in $y^2 + z^2 = 1$, we obtain $y^2 + 4y^2 = 1 \iff y = \pm\frac{1}{\sqrt{5}}$. Hence the two solutions are: $(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, and $(0, -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$.

Observe that

$$D\Big(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\Big) = 6 - 2\sqrt{5},$$

and that

$$D\Big(0, -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\Big) = 6 + 2\sqrt{5}.$$

Clearly, $D(0, -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}) > D(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, hence $(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ corresponds to the point of the minimum distance, and $(0, -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$ corresponds to the point of the maximum distance.

Problem 3: (a; 5 points) Change the order of integration and evaluate the following integral:

$$I = \int_0^9 \int_{\sqrt{x}}^3 \cos\left(y^3\right)\, dy\, dx.$$

We can rewrite the domain of integration like so:

$$D = \left\{0 \leq x \leq 9,\ \sqrt{x} \leq y \leq 3\right\} = \left\{0 \leq y \leq 3,\ 0 \leq x \leq y^2\right\}.$$

This allows us to rewrite and compute this integral:

$$I = \int_0^3 \int_0^{y^2} \cos\left(y^3\right) dx\, dy = \int_0^3 y^2 \cos\left(y^3\right) dy = \left[\tfrac{1}{3}\sin\left(y^3\right)\right]_{y=0}^{y=3} = \tfrac{1}{3}\sin 27.$$

(b; 5 points) Consider the helix C parametrized by $\mathbf{r}(t) := \left\langle \cos t, \sin t, t\right\rangle$, $0 \leq t \leq 2\pi$. Compute the following integral:

$$\int_C z\, ds.$$

We have $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$, so $|\mathbf{r}'(t)| = \sqrt{2}$. We therefore have:

$$I = \int_C z\, ds = \int_0^{2\pi} \sqrt{2}t\, dt = \left[\tfrac{1}{2}\sqrt{2}t^2\right]_0^{2\pi} = 2\sqrt{2}\pi^2.$$

Problem 4: Consider the region E *below* the cone $z = \sqrt{x^2+y^2}$, within the cylinder $x^2+y^2 = 16$, and above the xy-plane. Consider the following integral, which represents the volume of E:

$$I := \iiint_E 1\, dV.$$

(a; 5 points) Rewrite I as an iterated integral in Cartesian coordinates. (You do not have to evaluate this iterated integral.)

OK:

$$I = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{\sqrt{x^2+y^2}} 1\, dz\, dy\, dx.$$

(b; 5 points) Rewrite I as an iterated integral in cylindrical coordinates. (You do not have to evaluate this iterated integral.)

OK:

$$I = \int_0^4 \int_0^{2\pi} \int_0^r r\, dz\, d\theta\, dr.$$

Problem 5 (10 points): Consider the following vector field:

$$\mathbf{F}(x,y) := \left\langle y + 2xe^{x^2+y^2}, x + 2ye^{x^2+y^2} \right\rangle.$$

(a; 2 points of extra credit) Without computing a potential function, argue carefully for why $\mathbf{F}$ is conservative.

Let $P(x,y) = y + 2x\exp(x^2+y^2)$, and $Q(x,y) = x + 2y\exp(x^2+y^2)$. Evidently these are smooth functions on $\mathbb{R}^2$, they have partial derivatives of every order on $\mathbb{R}^2$. Moreover,

$$\frac{\partial P}{\partial y} = 1 + 4xy\exp(x^2+y^2) = \frac{\partial Q}{\partial x}$$

on all of $\mathbb{R}^2$, and since $\mathbb{R}^2$ is simply connected, $\mathbf{F}(x,y)$ must be a conservative vector field by Theorem 6 of 13.3

(b; 5 points) Find a potential function for $\mathbf{F}$, i.e. a function f satisfying $\mathbf{F} = \nabla f$. Let

$f(x,y)$ be a potential function. Then $P = f_x$ and $Q = f_y$. Thus

$$f(x,y) = \int P(x,y)\,dx + g(y) = xy + \exp(x^2+y^2) + g(y),$$

for some function $g(y)$. Partial differentiating the both sides the equation with respect y, we obtain

$$f_y(x,y) = Q(x,y) = x + 2y\exp(x^2+y^2) + g'(y) = x + 2y\exp(x^2+y^2).$$

This implies $g'(y) = 0$, which is if and only if $g(y) = K$ for some constant K. Hence

$$f(x,y) = xy + \exp(x^2+y^2) + K.$$

(c; 5 points) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve parametrized by $\mathbf{r}(t) := \langle \sqrt{\log t}\cos t, \sqrt{\log t}\sin t \rangle$, $\pi \le t \le \frac{3\pi}{2}$.

Since the vector field is conservative, we only need to care about the starting and the ending points of the path. Observe that

$$f(\mathbf{r}(t)) = \log t \sin t \cos t + \exp(\log t \sin^2 t + \log t \cos^2 t) + K = \frac{1}{2}\log t \sin 2t + t + K.$$

Here, used the identity $\exp(\log x) = x$.

By the fundamental theorem for line integrals,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(3\pi/2)) - f(\mathbf{r}(\pi)) \\ &= \left(\frac{1}{2}\log\frac{3\pi}{2}\sin 3\pi + \frac{3\pi}{2} + K\right) - \left(\frac{1}{2}\log\pi\sin 2\pi + \pi + K\right) = \frac{\pi}{2}.\end{aligned}$$