Solutions to Midterm 1 for MATH 226, section 39559

You have 50 minutes.

Name: **Date:**

Problem	Score
#1	/10
#2	/10
#3	/10
#4	/10
Total	/40

(Note that you can earn up to two points of extra credit in Problem 3, part c!)

Problem 1: Consider

$$f(x,y) = \cos(\pi x^3 + \pi y^2) + \sin(\pi x + \pi y^4),$$

where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

with $r \geq 0$ and $\theta \in [0, 2\pi)$.

(a; 5 points) Compute $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{3}$. (Recall $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.)

We compute $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ using the chain rule:

$$\begin{aligned}\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} &= (-3\pi x^2 \sin(\pi x^3 + \pi y^2) + \pi\cos(\pi x + \pi y^4))(\cos\theta) \\ &\quad + (-2\pi y \sin(\pi x^3 + \pi y^2) + 4\pi y^3 \cos(\pi x + \pi y^4))(\sin\theta),\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} &= (-3\pi x^2 \sin(\pi x^3 + \pi y^2) + \pi\cos(\pi x + \pi y^4))(-r\sin\theta) \\ &\quad + (-2\pi y \sin(\pi x^3 + \pi y^2) + 4\pi y^3 \cos(\pi x + \pi y^4))(r\cos\theta).\end{aligned}$$

When $r = 2$, $\theta = \frac{\pi}{3}$, we have $x = 1$, $y = \sqrt{3}$. Therefore $\pi x^3 + \pi y^2 = \pi + 3\pi = 4\pi$ and $\pi x + \pi y^4 = \pi + 9\pi = 10\pi$. Using this:

$$\begin{aligned}\frac{\partial f}{\partial r}\left(2, \frac{\pi}{3}\right) &= (0 + \pi) \cdot \frac{1}{2} + (0 + 12\sqrt{3}\pi) \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}\pi + 18\pi = \frac{37}{2}\pi, \\ \frac{\partial f}{\partial \theta}\left(2, \frac{\pi}{3}\right) &= (0 + \pi)(-\sqrt{3}) + (0 + 12\sqrt{3}\pi) \cdot 1 = -\sqrt{3}\pi + 12\sqrt{3}\pi = 11\sqrt{3}\pi.\end{aligned}$$

(b; 5 points) Approximate, using calculus, the value of f at $r = \frac{2001}{1000}$ and $\theta = \frac{1001\pi}{3000}$. You do not have to write your answer as a single fraction.

The linear approximation of f at $r = 2$, $\theta = \frac{\pi}{3}$ is:

$$L(r,\theta) = f\left(2, \frac{\pi}{3}\right) + \frac{\partial f}{\partial r}\left(2, \frac{\pi}{3}\right)(r - 2) + \frac{\partial r}{\partial \theta}\left(2, \frac{\pi}{3}\right)\left(\theta - \frac{\pi}{3}\right) = 1 + \frac{37}{2}\pi(r-2) + 11\sqrt{3}\pi\left(\theta - \frac{\pi}{3}\right).$$

Evaluating this at the given point, we get:

$$\begin{aligned}f\left(\frac{2001}{1000}, \frac{1001\pi}{3000}\right) \approx L\left(\frac{2001}{1000}, \frac{1001\pi}{3000}\right) &= 1 + \frac{37}{2}\pi \cdot \frac{1}{1000} + 11\sqrt{3}\pi \cdot \frac{\pi}{3000} \\ &= 1 + \frac{37\pi}{2000} + \frac{11\sqrt{3}\pi^2}{3000}.\end{aligned}$$

Problem 2: Consider the two planes

$$P_1 : x + y - z = 2, \qquad P_2 : 3x - 4y + 5z = 6.$$

(a; 2 points) Find normal vectors $\mathbf{n}_1, \mathbf{n}_2$ to these planes.

We can read normal vectors off by looking at the coefficients on x, y, and z in the defining equations:

$$\mathbf{n}_1 = \langle 1, 1, -1 \rangle, \qquad \mathbf{n}_2 = \langle 3, -4, 5 \rangle.$$

(b; 2 points) Let L be the line of intersection of these planes. Find a point on L. (Hint: set one of the coordinates equal to 0; this is the intersection of L with a coordinate plane.)

Let's set $z = 0$. Then the equations become $x + y = 2$, $3x - 4y = 6$. The first yields $y = -x + 2$; substituting that into the second equation, we have $3x - 4(-x + 2) = 6$, hence $7x - 8 = 6$, hence $x = 2$. Now $y = -x + 2$ yields $y = 0$. We conclude that $(2, 0, 0)$ lies on L.

(c; 4 points) Find a vector equation for L. (Hint: to find a point on both planes, try setting one coordinate to zero, then solving the equations of the planes for x and y.)

We need a vector parallel to L. L lies on both P_1 and P_2, hence L is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. It follows that $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to L. We compute this cross product:

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 3 & -4 & 5 \end{vmatrix} = \langle 5 - 4, -(5 + 3), -4 - 3 \rangle = \langle 1, -8, -7 \rangle.$$

A vector equation for L is therefore given by:

$$\mathbf{p} = \langle 2, 0, 0 \rangle + t\langle 1, -8, -7 \rangle.$$

(d; 2 points) Find the angle between these planes.
We compute the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$:

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-6}{\sqrt{3} \cdot 5\sqrt{2}}\right) = \cos^{-1}\left(-\frac{\sqrt{6}}{5}\right).$$

This angle is greater than $\frac{\pi}{2}$ (because the dot product is negative). So the angle between the planes is given by:

$$\pi - \cos^{-1}\left(\frac{\sqrt{6}}{5}\right).$$

Problem 3: Consider the vector function $\mathbf{r}(t) := \left\langle \frac{2}{3}t^{3/2}, \cos t, \sin t \right\rangle$, $t \geq 0$.

(a; 5 points) Compute the arclength of the curve traced out by $\mathbf{r}$ from $t = 0$ to $t = 1$.

First, we compute $|\mathbf{r}'(t)|$:

$$|\mathbf{r}'(t)| = \left|\langle t^{1/2}, -\sin t, \cos t\rangle\right| = \sqrt{t + \sin^2 t + \cos^2 t} = \sqrt{t+1}.$$

We use this to compute the arclength:

$$L = \int_0^1 |\mathbf{r}'(u)|\, du = \int_0^1 \sqrt{u+1}\, du = \left[\frac{2}{3}(u+1)^{3/2}\right]_{u=0}^{u=1} = \frac{4\sqrt{2}-2}{3}.$$

(b; 5 points) Compute the reparametrization of $\mathbf{r}(t)$ by arclength, starting from $t = 0$ and in the direction of increasing t.

The arclength is given by the following formula (picking up from the previous part):

$$s(t) = \int_0^t |\mathbf{r}'(u)|\, du = \frac{2}{3}(t+1)^{3/2} - \frac{2}{3}.$$

We solve for t in terms of s:

$$\begin{aligned} s = \frac{2}{3}(t+1)^{3/2} - \frac{2}{3} \implies \frac{2}{3}(t+1)^{3/2} = s + \frac{2}{3} &\implies (t+1)^{3/2} = \frac{3}{2}s + 1 \\ &\implies t + 1 = \left(\frac{3}{2}s+1\right)^{2/3} \\ &\implies t = \left(\frac{3}{2}s+1\right)^{2/3} - 1. \end{aligned}$$

We now write out $\mathbf{r}(t(s))$, which is the reparametrization of $\mathbf{r}(t)$ by arclength:

$$\begin{aligned} \mathbf{r}(t(s)) &= \mathbf{r}\left(\left(\frac{3}{2}s+1\right)^{2/3} - 1\right) \\ &= \left\langle \tfrac{2}{3}\left(\left(\tfrac{3}{2}s+1\right)^{2/3} - 1\right)^{3/2}, \cos\left(\left(\tfrac{3}{2}s+1\right)^{2/3} - 1\right), \sin\left(\left(\tfrac{3}{2}s+1\right)^{2/3} - 1\right)\right\rangle. \end{aligned}$$

(c; 2 points of extra credit) Draw a picture of the curve traced out by $\mathbf{r}$, with justification.

Note that the projection of $\mathbf{r}(t)$ to the yz-plane is $\langle 0, \cos t, \sin t\rangle$, which is a parametrization of the unit circle. The first component $x(t) = \frac{2}{3}t^{3/2}$ increases from $x = 0$ to $x = +\infty$ as t ranges over the nonnegative reals. Therefore we get a half-helix centered on the positive x-axis, where the distance between spirals increases as we move farther from the origin.

Problem 4: A series of small questions. Make sure to justify, in a sentence or two.

(a; 2 points) True or false: the cross product of two 3-dimensional vectors is a scalar.

False! It's a 3-dimensional vector.

(b; 2 points) True or false: If $\mathbf{v}, \mathbf{w}$ are any vectors of the same dimension, then $\mathbf{v} \cdot \mathbf{w}$ makes sense and is zero exactly when $\mathbf{v}$ and $\mathbf{w}$ are orthogonal (i.e., perpendicular).

True! This is an immediate consequence of the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$.

(c; 2 points) True or false: the z-slices of the quadric surface $x^2 + y - 2z^2 = 1$ are ellipses.

False! When we set $z = c$, this equation becomes $y = -x^2 + 2c^2 + 1$, which is a parabola.

(d; 2 points) Suppose that we are standing at the base of a mountain. We are standing at the origin $(0,0)$, and the mountain is located at $(0,1)$ (due north). Let $E(x,y)$ denote the elevation at the point (x,y). What can you say about the sign of $\frac{\partial E}{\partial y}(0,0)$?

This partial derivative represents the rate of change as we hold x constant and increase y. The elevation is increasing as we move in the direction of increasing y, so this partial derivative is positive.

(e; 2 points) True or false: if $f(x,y) = x^2$, then $\nabla f(x,y) = 2x$.

False! The gradient is a vector, and in this case we have $\nabla f(x,y) = \langle 2x, 0 \rangle$.