Math 226/229 General Final Review

Chapter 10: Vectors and the Geometry of Space

Important Formulas

Distance Formula: $\sqrt{(\Delta x)^2 + (\Delta y) + (\Delta z)^2}$

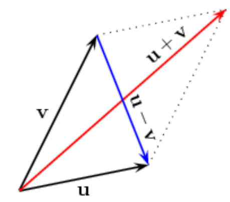
Sphere: $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Vector Magnitude: $|\mathbf{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

Unit Vector: $\hat{\mathbf{u}} = \dfrac{\mathbf{u}}{|\mathbf{u}|}$

Vector Multiplied by a Scalar:
$k*\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle$

Vector Addition/Subtraction:



Dot Product: $\mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3 = |\mathbf{v}||\mathbf{w}|\, cos(\theta)$

Scalar Projection: $\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{a \bullet b}{|a|}$

Vector Projection: $\text{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{a \bullet b}{|a|^2}a$

Cross Product: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

Magnitude: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\, sin(\theta)$

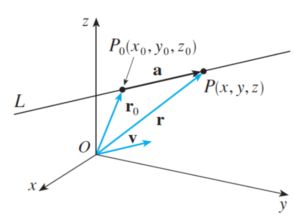
Triple Product: Volume of parallelipiped = $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$

Lines

Vector Form: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

Parametric Form: $x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$

Symmetric Form: $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$



Planes

Vector Form: $\mathbf{n} \bullet (\mathbf{r} - \mathbf{r}_0) = 0$

Scalar Form: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \Rightarrow ax + by + cz = d$

Arc Length: $L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$

Unit Tangent Vector: $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Some Things to Keep in Mind

- Two vectors are orthogonal iff **v** • **w** = 0
- Two vectors are parallel if **v** is a scalar multiple of **w** (or **v** × **w** = 0)
- **v** × **w** is orthogonal to both **v** and **w**
- Three vectors are coplanar iff their triple product is 0
- To write the equation for a line, you need a **point** and a **direction** vector
- To write the equation for a plane, you need a **point** and a **normal** vector

Tips for Parameterizing Curves

- Rewrite equations isolating one variable in terms of another, then substitute for t
- Polar coordinates are common for shapes like ellipses and circles ($x = a\ cos(t)$, $y = b\ sin(t)$)
- Curves can be parametrized in an infinite number of ways, but some will make your calculations easier than others

Sketching Quadric Surfaces

- If the equation only has two variables, draw the curve on those two axes, and then stretch the image along the remaining axis to obtain the 3D shape.
  - Ex. Sketch the surface $4x^2 + y^2 = 16$
- Draw traces of the graph by holding one variable constant and graphing the remaining two variables in two dimensions. Repeat until you can determine the shape of the graph. You may need to change which variable is held constant to get a clear picture of the shape.
  - Ex. Sketch the surface $y = z^2 + x^2$

Practice Problems

1. UConn Fall 2008 Midterm 1 Q1
    a. Give a vector equation of the line containing the points $(1, -3, 2)$ and $(4, 1, 0)$.

    b. Find the intersection of the line with the plane $x + y + z = 10$.

    c. Is the point $(10, 9, -4)$ on this line?

2. USC Fall 2016 Final Q1

   Consider two planes, P_1: $x + y = 27$ and P_2: $2x + z = 10$, and a point $Q = (3, 4, 1)$.

   a. Write an equation of the plane that passes through the point Q and is perpendicular to the planes P_1 and P_2.

   b. Write a parametric equation of the line that passes through the point Q and is parallel to the planes P_1 and P_2.

3. UConn Fall 2008 Midterm 1 Q2

For the curve $\mathbf{r}(t) = \langle 2t^3,\ 1 - t^3,\ -2t^3 \rangle,\ t \geq 0$:

a. Reparametrize the curve with respect to arc-length, starting at $t = 0$ and moving in direction of increasing t.

b. Find the distance along the curve from $(2, 0, -2)$ to $(16, -7, 16)$.

4. Colgate Spring 2003 Practice Exam 1 Q7

Let $\mathbf{v} = 2\,\mathbf{i} + a\,\mathbf{j} + a^2\,\mathbf{k}$ and $\mathbf{w} = (2a - 3)\,\mathbf{i} + \mathbf{j} + \mathbf{k}$:

a. For which values of a are the vectors perpendicular?

b. Write $\mathbf{v}$ as the sum of a vector parallel to $\mathbf{w}$ and a vector perpendicular to $\mathbf{w}$ when $a = 0$ (Hint: use vector projection!).

Chapter 11: Partial Derivatives
Important Formulas

Tangent Plane: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Linear Approximation: $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Total Differential: $dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$

Implicit Differentiation:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Gradient:

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Directional Derivative:

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Second Derivatives Test: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$

- If D > 0 and $f_{xx} > 0$, $f(a,b)$ is a local minimum
- If D > 0 and $f_{xx} < 0$, $f(a,b)$ is a local maximum
- If D < 0, $f(a,b)$ is a saddle point

System of Equations for Critical Points on a Constraint:

$$g(x, y, z) = k \qquad \nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

Some Things to Keep in Mind:

- The gradient vector shows the direction of the maximum increase of a function
- The gradient vector is normal to level surfaces and contour lines
- Beware of divide by zero errors when solving Lagrange multiplier problems!

Practice Problems

5. Colgate Spring 2003 Practice Exam 2 Q10
   Suppose f is a differentiable function such that

$$f(1,3) = 1, \qquad f_x(1,3) = 2, \text{ and } f_y(1,3) = 4.$$

   a. Find a vector in the plane that is perpendicular to the contour line $f(x,y) = 1$ at the point (1, 3).

   b. At the point (1, 3), what is the rate of change of f in the direction $\mathbf{i} + \mathbf{j}$?

6. MIT OpenCourseWare
   Let $w = xyz,\ x = u^2v,\ z = u^2 + v^2$. Find $\frac{\partial w}{\partial u}$ at the point $(u,v) = (1,2)$.

7. UConn Fall 2008 Midterm 2 Q2
   Consider the surface

$$2x^2y^5 + (zx)^2 - x^4yz = 16$$

   a. Find the equation of the tangent plane to this surface at the point (2, 0, 2).

   b. Approximate the y value of the point on the surface where $x = 2.03$ and $z = 1.99$.

8. MIT OpenCourseWare
   Find the maximum and minimum values of $f(x, y) = x^2 + x + 2y^2$ on the unit circle.

9. USC Spring 2015 Final Q3

Consider the function f that is given by

$$f(x, y) = y^3 - 2xy + x^2$$

a. Find all the critical points of f.

b. Classify each critical point you found above (if you can) as a local maximum, local minimum, or saddle point.

c. Does f have a *global* maximum on the plane? If it does, state its value and where it is attained. If not, explain why not.

d. Does f have a *global* minimum on the plane? If it does, state its value and where it is attained. If not, explain why not.

12.1: Double Integrals Over Rectangles

Iterative integrals can be applied to computing volumes for solids in 3 dimensions. We can define the bounds of a rectangle R = [a, b] (bounds of x) and [c, d] (bounds of y) as a rectangle R. The function f :R→ $\mathbb{R}$ is considered the height of the solid $f(x, y)$. We can determine the volume of the solid bounded below by the rectangle R and above by the function f by double integration:

$$\iint_R f(x,y)dA = \iint_R f(x,y)dxdy$$

We can think of this double integral similar to a double Riemann sum from Calculus I, where we can approximate the volume under the graph of f by rectangles.
We can compute double integrals as iterated integrals. Using the above example to calculate the volume of the solid under f, we can compute:

$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_c^d f(x,y)dy \right] dx$$

An important property of double integrals is **Fubini's theorem** - that is, we can change the order of integration and the result will be equivalent. This is especially helpful when one of the integrals is too difficult to be computed by hand. By switching the order, it's possible to produce an easier integral to evaluate.

$$\iint_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy \right] dx = \int_c^d \left[\int_a^b f(x,y)dx \right] dy$$

It is also possible to separate out the integrals if the functions are not dependent on each other. For example, if we can write $f(x, y)$ as $g(x)$ and $h(y)$, then:

$$\iint_R f(x,y)dA = \iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy$$

12.2: Double Integrals Over General Regions

Double integrals can be extended to more general regions in the Cartesian plane. There are two types of general regions over which we can integrate:

1. Type 1 region: the region is bounded on the left and ride by $x = a$ and $x = b$ respectively, and bounded on the bottom and top by $y = g_1(x)$ and $y = g_2(x)$ respectively. Notice that this is exactly the same as rectangular regions, but we now assume that $y = g(x)$ can be any function, and not just a straight line. This can computed as follows:

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\ dx$$

2. Type 2 region: similar, but the region is now bounded on the left and right by $x = h_1(y)$ and $x = h_2(y)$ respectively, and bounded on the bottom and top by $y = c$ and $y = d$ respectively. We change the order of integration in this case:

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\ dy$$

We can also apply Fubini's theorem to computing integrals in a different order.

12.3 : Double Integrals in Polar Coordinates

Our regions aren't restricted to functions of x and y; sometimes it is easier to describe the region of interest by its bounds in polar coordinates in order to compute its double integral. Recall 3 simple equations we can use to convert rectangular coordinates to polar coordinates, and vice versa:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

where r represents the radius/distance from a point to the origin and θ represents the angle between r and the x-axis. So we can write our rectangular points $P(x, y)$ as $P(r, \theta)$. This changes our bounds of integration and we rewrite our function of integration in terms of $r\cos\theta$ and $r\sin\theta$ for x and y respectively:

$$dA = r\ dr\ d\theta = r\ d\theta\ dr$$

$$\iint_R f(x, y)\ dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\mathbf{r}\ d\theta dr = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\mathbf{r}\ drd\theta$$

where our bounds for r are the constants (a, b) and our bounds for θ are the angles (α, β) in radians.

Don't forget the Jacobian factor r!

12.4: Applications of Double Integrals

There are many physics-based applications of double integrals that extend beyond simply computing volumes of solids in 3 dimensions.

Density and Mass: we can determine the mass of an object if we integrate the density function of the object over a general region, as such:

$$m = \iint_D \rho(x,y)\, dA$$

where the density ρ is represented as a function of the point (x,y).

Moments and Center of Mass: we can also calculate the individual moments about the x-axis and y-axis in order to determine the center of mass of a lamina with variable density (the density is not constant throughout):

$$M_x = \iint_D y\rho(x,y)\, dA, \quad M_y = \iint_D x\rho(x,y)\, dA$$

to compute the moments about the x and y axis respectively. From these double integral we can calculate the center of mass $(\overline{x}, \overline{y})$ of the lamina over the region D with density function $\rho(x,y)$ and mass m:

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)\, dA, \quad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)\, dA$$

Finally, we can compute the moments of inertia of lamina/disk about the x- and y-axis respectively:

$$I_x = \iint_D y^2\rho(x,y)\, dA, \quad I_y = \iint_D x^2\rho(x,y)\, dA$$

and the polar moment of inertia/moment of inertia about the origin is:

$$I_0 = \iint_D (x^2+y^2)\rho(x,y)\, dA$$

where

$$I_0 = I_x + I_y$$

12.5: Triple Integrals

The same way we define integrals of single and double variable functions, we can do the same for three variable functions. Let's define a simple box B as the product of x, y, z. If we want to take the triple integral of the function f over a bounded region in 3 dimensions E, then we define this integral as:

$$\iiint_E f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dxdydz$$

for the box B defined as $[a,b] \times [c,d] \times [e,f]$.

If we can write each of three functions as a function of one variable, then we can separate them into three individual single integrals:

$$\iiint_E f(x)g(y)h(z)\, dV = \int_r^s h(z)\, dz \int_c^d g(y)\, dy \int_a^b f(x)\, dx$$

Additionally, properties of double and single integrals apply to triple integrals. We can change the order of the integrals (6 orders of integration) and take constants out of the integral.

More generally, we can write two of the three variables of integration as functions of other variables. This is what we did for Type I and Type II regions for double integrals ($f_(y),\ g(x)$) for the bounds of the region. Now, we have a bounded region in $\mathbb{R}^{\not\models}$ where we can write our bounds of integration in terms of the other variables:

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\, dzdydx$$

is one order of integration. Notice that the bounds for z are functions of x, y, the bounds for y are functions of x, and the bounds for x are constants. This is the most challenging part of triple integral problems: setting up the bounds of integration over the region E.

Perhaps the above order is difficult to compute, so we can rearrange the order of integration to describe the region.

12.6: Triple Integrals in Cylindrical Coordinates

Recall our conversion of rectangular coordinates to polar coordinates in 12.3. We can extend this system to triple integrals by use of cylindrical coordinates:

$$x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$$

To convert from rectangular to cylindrical, we can use:

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

What would our triple integral look like in cylindrical coordinates? Suppose we can describe our region E conveniently in cylindrical coordinates for our function $f(x, y, z)$. We would integrate over z, r, θ respectively, as such:

$$\int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\ \mathbf{r}\ dzdrd\theta$$

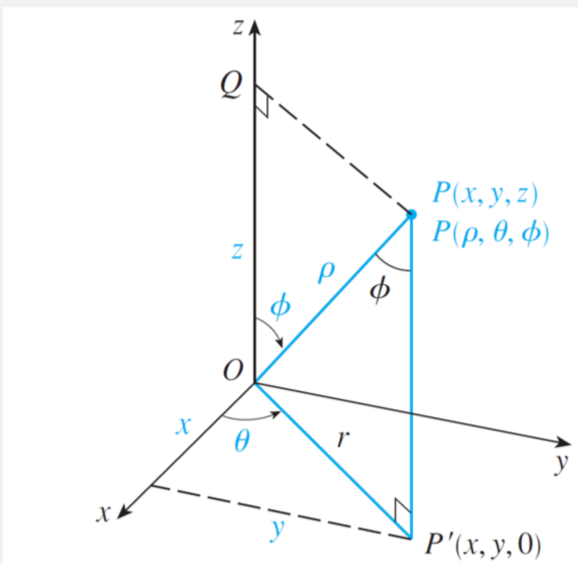
Note that we again use a Jacobian factor r when computing the triple integral in cylindrical coordinates, based on the polar coordinate system. The bounds of the integration are θ over $[\alpha, \beta]$, r over $[h_1(\theta), h_2(\theta)]$, and z over $[u_1(r\cos\theta, r\sin\theta), u_2(r\cos\theta, r\sin\theta)]$.

That's it! We can apply similar techniques in converting from rectangular coordinates to polar coordinates for double integrals to using cylindrical coordinates for triple integrals.

12.7: Spherical Coordinates

We now arrive at the last coordinate system for iterative integration. So far we've covered rectangular coordinates in double and triple integrals, polar coordinates in double integrals, and cylindrical coordinates in triple integrals.

Now we cover spherical coordinates in triple integrals. Spherical coordinates are the most complex, and we define a whole new set of equations to convert from rectangular to spherical coordinates.



Here we define spherical coordinates (ρ, ϕ, θ) using the following equations:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

for $0 \leq \phi \leq \pi$ and $\rho \geq 0$. From these equations we can also derive:

$$x^2 + y^2 + z^2 = \rho^2$$

And when we evaluate triple integrals, we have

$$\iiint_E f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, dV$$

13.1: Vector Fields

In this section we introduce the concept of vector fields, and from here we can build upon our foundational knowledge of vector fields to apply essential vector calculus theorems that you may have heard of (Stokes Thm, Green's Thm, etc) to compute integrals.

Essentially, a vector field is a function that assigns a point in space a vector (magntiude and direction) that we denote $\mathbf{F}$. For example, we can have a velocity or force vector field in which each point (x, y) is assigned a velocity or force vector.

We defined the gradient vector ∇f in Chapter 11. If we let

$$f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

then we can let ∇f be the gradient vector field of f. Additionally, we can define a **conservative** vector field if there exists a function f such that

$$\mathbf{F} = \nabla f$$

in other words, the vector field $\mathbf{F}$ is the gradient of some function f. We define f for conservative vector fields as the potential function of $\mathbf{F}$.

13.2: Line Integrals

Previously we defined integrals over general regions over some interval $[a, b]$. Now we can extend integrals to apply to smooth, parameterized curves that are defined as some function of a parameter e.g. t.

For some curve C, we can only take its line integral if it is smooth and parameterized with respect to some one-to-one vector function $\mathbf{r}(\mathrm{t})$. The curve can only be smooth if $r'(t)$ is continuous on the interval $[a, b]$.

If f is a continuous function on a smooth curve C, then the line integral of f over C is:

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\, \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt = \int_a^b f(\mathbf{r}(t))|\mathbf{r'}(t)|\, dt$$

Recall that the second part of the integral is the length of the curve C that we defined in Chapter 10. The line integral is not dependent on the paramterization of the curve.

We can also define line integrals in 3D space, exactly the same as we define for the Cartesian plane with one extra variable:

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\, \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt = \int_a^b f(\mathbf{r}(t))|\mathbf{r'}(t)|\, dt$$

Finally, we can define line integrals in terms of vector fields: for some vector field $\mathbf{F}$ and smooth curve C, the line integral of $\mathbf{F}$ along C is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

where $\mathbf{F}(\mathbf{r}(t))$ is simply $\mathbf{F}(x(t), y(t), z(t))$.

13.3: FTC for Line Integrals

Suppose we have a smooth curve C parameterized by a vector function $\mathbf{r}$(t). If f is differentiable and its gradient ∇f is on C, then:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

which essentially says that we can compute a line integral of a conservative vector field with only the endpoints of the curve. If the curve C is **closed**, then the above line integral evaluates to 0.

One important theorem of the FTC for line integrals is that of path independence: for a field $\mathbf{F}$ on domain D, $\mathbf{F}$ is path independent if and only if

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every closed curve C in D. Additionally, if $\mathbf{F}$ is path independent, then we can say that $\mathbf{F}$ is a conservative vector field s.t. a potential function exists: $\nabla f = \mathbf{F}$.

We can determine whether a vector field is conservative or not as well, by the following theorems. $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a vector field on a simply-connected region D. If the following partial derivatives are defined:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ throughout } D$$

then $\mathbf{F}$ is conservative.

13.4: Green's Theorem

This is the first of our big three vector calculus theorems! We need only define Green's Theorem here. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P$and Q have continuous partial derivatives on an open region that contains D, then:

$$\int_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

where

$$\mathbf{F} = < P, Q >$$

13.5: Curl and Divergence

Curl and divergence are basic operations on vector fields that are essential for understanding problems in physics and engineering. Let's say we have a vector field **F** defined by its x, y, z components P, Q, and R respectively, each with first-order partial derivatives. Then we can define the curl of the vector field **F** as:

$$\text{curl}\,\mathbf{F} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\,\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\,\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\,\mathbf{k}$$

Note that this is a vector function such that the curl of the vector field **F** is always a vector.

An important operator to know is the differential "del" operator ∇, defined as follows:

$$\nabla = \frac{d}{\partial x}\,\mathbf{i} + \frac{d}{\partial y}\,\mathbf{j} + \frac{d}{\partial z}\,\mathbf{k}$$

We can relate the del operator to curl by taking the cross of ∇ and **F** to produce the curl:

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

Another key theorem is that the curl of a gradient vector field is 0:

$$\text{curl}(\nabla f) = \mathbf{0}$$

Thus, it follows that a conservative vector field will have a curl of **0**, since there exists a potential function such that $\mathbf{F} = f$. Recall from the earlier section on conservative vector fields that, for a 2-D vector field, we would simply need to show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

We simply extend this relationship to three dimensions when we work with a vector field with P, Q,and R components.

We can understand curl in the context of rotations and fluid flow. If we have a vector field **F** that represents the velocity of a fluid, then particles at (x, y, z) tend to rotate about the axis that points in the direction of the curl. If the curl is 0 (conservative), then the fluid is free from rotations at that point.

Now we define the divergence of a vector field as a new operator, which differs from the curl in that it is a scalar function:

$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

div $\mathbf{F}$ is a scalar, not a vector field, which can also be computed as a dot product:

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

One key property of divergence:

$$\text{div curl}\,\mathbf{F} = 0$$

for all vector fields.

Divergence can be understood in the context of fluid flow: if we have a vector field $\mathbf{F}$ that represents the velocity of a liquid, the divergence of $\mathbf{F}$ gives the net rate of change of the mass of fluid flowing from a point (x, y, z) per unit volume, or the tendency of the fluid to diverge from that point.

13.6: Parametric Surfaces

Recall that we can parameterize a space curve by a vector function $\mathbf{r}(\mathrm{t})$. We can extend this parameterization to surfaces in 3 dimensions, such that we have a function

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$$

that is a function from the region D on the uv plane to $\mathbb{R}^3$. By using $\mathbf{r}(u,v)$, we can say that a surface S is a parameterized surface defined by $\mathbf{r}(u,v)$, which maps the uv plane to a surface in $\mathbb{R}^3$.

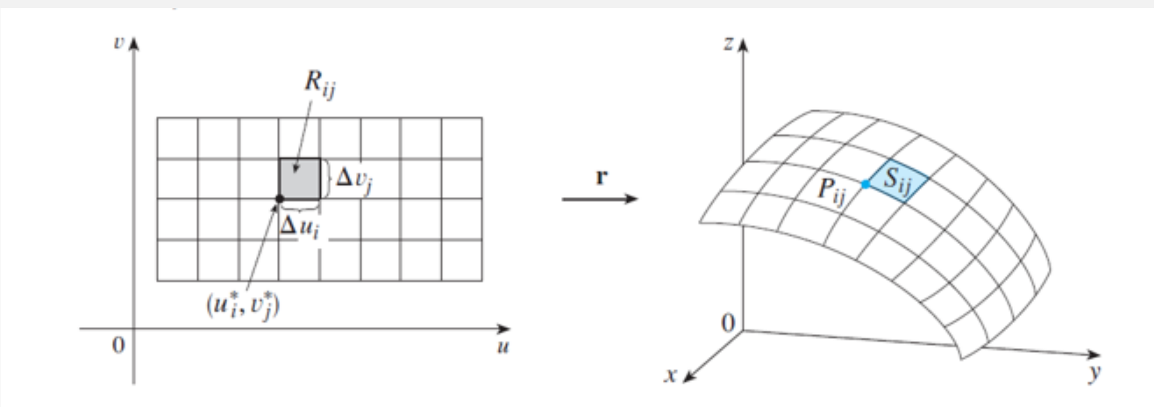
Additionally, we can define tangent vectors and tangent planes in 3 dimensions. If the parameterization of S is defined by $r(u,v)$, then the tangent vectors of S are:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{x}}{\partial u}\,\mathbf{i} + \frac{\partial \mathbf{y}}{\partial u}\,\mathbf{j} + \frac{\partial \mathbf{z}}{\partial u}\,\mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{x}}{\partial v}\,\mathbf{i} + \frac{\partial \mathbf{y}}{\partial v}\,\mathbf{j} + \frac{\partial \mathbf{z}}{\partial v}\,\mathbf{k}$$

Given the tangent vectors to S at some point (u_0, v_0), we can find the tangent plane through that point by using the cross product of r_u and r_v.

Finally, we can use the parameterization of a surface S to compute its surface area.



The area of patch on the surface on the right side can be approximated by taking the magnitude of the cross product of its tangent vectors on the Cartesian plane:

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

for a region D in $\mathbb{R}^2$ and paramterization $\mathbf{r}(u,v)$.

Now let's say we want to find the surface area for the graph of a function:

$$x = x, \quad y = y, \quad z = f(x, y)$$

If we take $|r_x \times r_y|$, we will see that this simplifies to

$$\sqrt{1 + f_x^2 + f_y^2}$$

So we have

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

13.7: Surface Integrals

If we let S be a smooth surface parameterized by $\mathbf{r}(u,v)$, and define $f(x,y,z)$ as a continuous function on S. Here we can define the integral of f over S as

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

Now let's say we have a graph $g(x,y)$ parameterized by

$$x = x, \quad y = y, \quad z = g(x,y)$$

Then we can rewrite the surface integral as

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{1 + f_x^2 + f_y^2}\,dA$$

For graphs, we can also define oriented surfaces and the associated normal vector. A **positively** oriented closed surface is one in which its normal vectors point outward (away from the surface). We can define the unit normal vector as

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + g_x^2 + g_y^2}}$$

Thus it follows that the normal vector in the opposite direction is **-n**.

Finally, and most importantly, we have surface integrals across vector fields. Let the surface S have parameterization $\mathbf{r}(u,v)$ defined on D and the vector field $\mathbf{F}$ is defined on S. Then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \cdot dA$$

Also defined as the flux of $\mathbf{F}$ across S.

13.8: Stokes' Theorem

Stokes' Theorem is actually simply a generalization of Green's Theorem, which we studied earlier. Instead of relating the double integral over a plane region to a line integral over its boundary curve, we now relate the surface integral over the surface S to the line integral of the boundary curve around S.

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive (outward) orientation. Let $\mathbf{F}$be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot dr = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

13.9: Divergence Theorem

The Divergence Theorem is similar to Stokes' Theorem in that it also relates an integral of the derivative of a given function over a region to the integral of the function over its boundary. We can generalize this theorem to higher dimensions.

Let's define a simple solid region E such that it is a closed surface with positive/outward orientation (the normal vector $\mathbf{n}$ points away from the region). If we let a surface S be the boundary of the region E with outward orientation, and we define a vector field $\mathbf{F}$ whose continuous partials exist, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E div \, \mathbf{F} \, dV$$

Let E be the region consisting of those points (x, y, z) in the first octant (where $x \geq 0, y \geq 0, z \geq 0$) and under the paraboloid of equation $z = 2x^2 2y^2$. Find the bounds of integration in the following order:

$$\iiint_E f(x, y, z) = \iiint f(x, y, z)\, dydxdz$$

Evaluate the following integral:

$$\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)\, dxdy$$

Evaluate the following integral:

$$\iiint_E x^2 \, dV$$

where E is the solid bounded by $y = \sqrt{9 - x^2 - z^2}$ and $y = \sqrt{16 - x^2 - z^2}$ and the xz plane.

Suppose a particle of mass M is located at the origin and a particle of mass m is located at the point (x, y, z). The gravitational force between the two particles is given by

$$\mathbf{F} =< \frac{-mMgx}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMgy}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMgz}{(x^2+y^2+z^2)^{3/2}} >$$

- Show that $\mathbf{F}$ is conservative
- Find the potential function of $\mathbf{F}$
- A second particle moves from (1,2,3) to (2,1,0) along the curve C parameterized by

  $$\mathbf{r}(t) =< t+e^t, 1+\sin(\frac{\pi t}{2}), 3-3t >$$

  for $0 \leq t \leq 1$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$

Evaluate

$$\oint_C (x^4)dx + (xy)dy$$

where C is given by the lines from (1,0) to (2,0), from (2,0) to (0,1), and from (0,1) to (1,0).

Consider the surface S with upwards orientation defined as the open paraboloid $z = 4 - x^2 - y^2$ for $3 \leq z \leq 4$. Find the surface area of S.

Use any method to find the line integral:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = y\,\mathbf{i} + x^2\,\mathbf{j} + xz^3\,\mathbf{k}$, and C is the triangle with vertices (0, 0, 3), (2, 0, 3) and (0, 8, 3), oriented by the ordering of the points.

Let E be the three-dimensional region cut from the first octant by the cylinder $x^2+y^2=4$ and the plane $z=3$, and let S be the bounding surface of E. The vector field $\mathbf{F}$ is:

$$\mathbf{F} = (6x^2 + 2xy)\,\mathbf{i} + (2y + x^2 z)\,\mathbf{j} + (4x^2y^3)\,\mathbf{k}$$

Find the flux of $\mathbf{F}$ out of the surface S:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

Good luck with the final exam, you got this!!

Food place of the week! - Din Tai Fung!