OH: today, noon - 1.

§10.7: Vector functions and space curves.

A vector function $\underline{r}(t)$ has domain $U \subset \mathbb{R}$. "is a subset of"

target $\mathbb{R}^n$.

Motivating example: $\underline{r}(t)$ is the position of a particle at time t.

$$\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$$
$$= x(t)\,\underline{i} + y(t)\,\underline{j} + z(t)\,\underline{k}.$$

"component functions"

Eg., $\underline{r}(t) := \left\langle \frac{1-t}{\sqrt{t-1}}, \log(2-t), 2 \right\rangle.$

need $t-1>0 \iff t>1$

need $2-t>0 \iff t<2$

domain is $1<t<2$

$(1,2)$

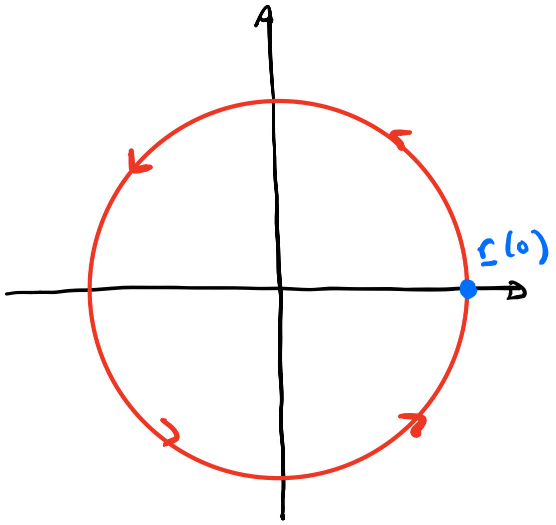
Important ex #1. $\underline{r}(t) := \langle \cos t, \sin t \rangle$. ($x(t)$, $y(t)$)

general strategy: find equation that components satisfy.

Here, $\cos^2 t + \sin^2 t = 1 \implies x(t)^2 + y(t)^2 = 1$

$$\implies |\underline{r}(t)| = \sqrt{x(t)^2 + y(t)^2} = 1.$$

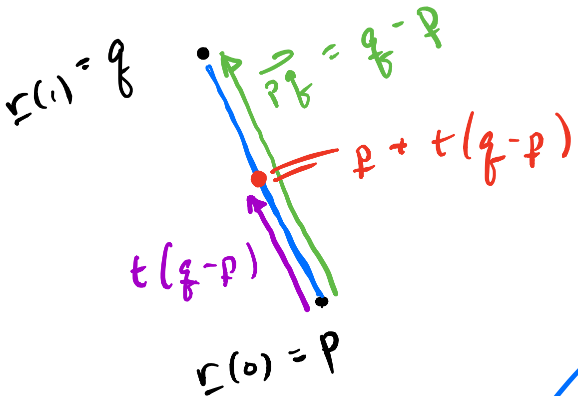
$\mathbb{R}^2$



$\underline{r}(0) = \langle 1, 0 \rangle.$

"are elements of"

Important ex #2. Say $p, q \in \mathbb{R}^3$. How to cook up $\underline{r}(t)$ that parametrizes line segment from p to q?

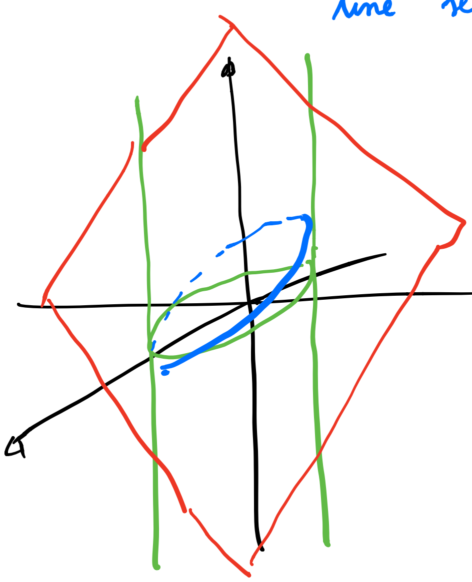


$$\boxed{\underline{r}(t) := p + t(q-p), \quad 0 \le t \le 1 \\ = (1-t)p + tq, \quad 0 \le t \le 1.}$$

$$\underline{r}(0) = 1 \cdot p + 0 \cdot q = p$$

$$\underline{r}(1) = 0 \cdot p + 1 \cdot q = q.$$

general way to parametrize line segment from p to q.



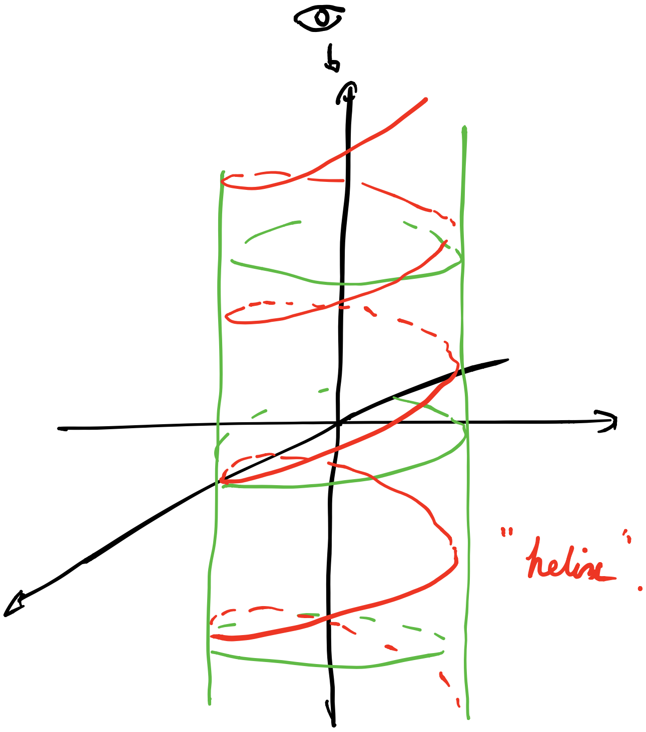
$\langle 2\cos t, \ \sin t, \ \cos t + \sin t \rangle$

$\left(\frac{1}{2}x\right)^2 + y^2 = 1, \quad \frac{1}{2}x + y = z.$

Ex 4. Describe the curve $\underline{r}(t) := \langle \cos t, \sin t, t \rangle$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x(t) \quad y(t) \quad z(t)$

$t \in \mathbb{R}$

note, $x(t)^2 + y(t)^2 = 1 \implies$ image of $\underline{r}(t)$ lies on the cylinder $\underline{r}(t)$.



Also, if we project to xy-plane, we get $\langle \cos t, \sin t, 0 \rangle$.

$\triangle$

Derivatives. $\quad \underline{r}'(t) := \lim_{h \to 0} \dfrac{\underline{r}(t+h) - \underline{r}(t)}{h}$.

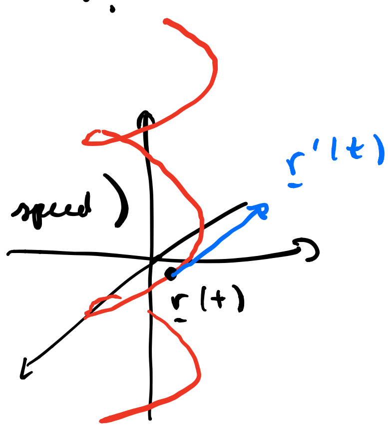
$$= \langle f'(t), g'(t), h'(t) \rangle.$$

Motivation:

$\underline{r}(t)$ position

$\underline{r}'(t)$ velocity $\quad (|\underline{r}'(t)|$ is speed$)$

$\underline{r}''(t)$ acceleration



Spring '12, 2a. $\underline{r}(t) := \langle t, t^2, t^3 \rangle$. Velocity @ $t=2$?
speed @ $t=2$?

$\underline{r}'(t) = \langle 1, 2t, 3t^2 \rangle \Rightarrow \underline{r}'(2) = \langle 1, 4, 12 \rangle$.

velocity

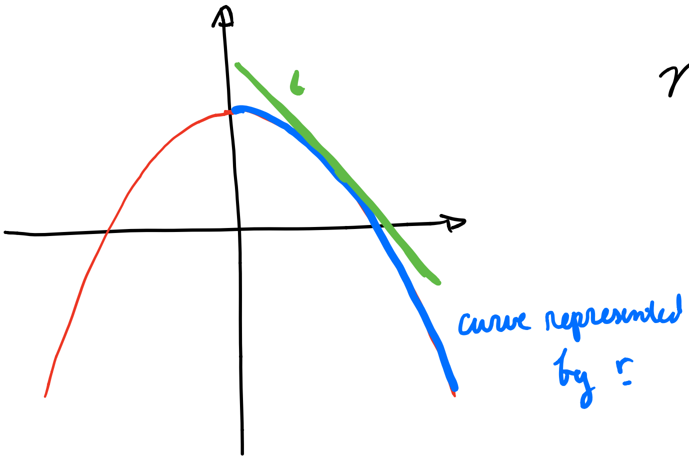
$\underline{r}'(t)$ is a tangent vector to $\underline{r}(t)$. A unit tangent vector to the curve defined by $\underline{r}$, at time t, is $\boxed{\dfrac{\underline{r}'(t)}{|\underline{r}'(t)|}}$.

[OH]

§10.7: Vector functions and space curves. (cont')

~ Ex 9: $\underline{r}(t) := \langle \underbrace{\sqrt{t}}_{x(t)}, \underbrace{2-t}_{y(t)} \rangle$. Describe image of $\underline{r}$ in terms of equations, and find tangent line @ $\langle 1,1 \rangle$.

A. $2 - x(t)^2 = 2 - t^2 = y(t)$. So the image curve lies in the curve $y = 2 - x^2$. But only the portion with $x \geq 0$!



Now, what's the t_0 w/ $\underline{r}(t_0) = \langle 1,1 \rangle$?

$\underline{r}(t_0) = \langle 1,1 \rangle \Leftrightarrow \langle \sqrt{t_0}, 2 - t_0 \rangle = \langle 1,1 \rangle$

$\Leftrightarrow t_0 = 1.$

Need a point on l and a vector tangent to it.
(1,1)

$\underline{r}'(1)$. $\underline{r}'(t) = \langle \frac{1}{2} t^{-1/2}, -1 \rangle$

$\Rightarrow \underline{r}'(1) = \langle \frac{1}{2}, -1 \rangle$.

So the line l is $\boxed{x = \frac{1}{2}t + 1, \quad y = -t + 1.}$

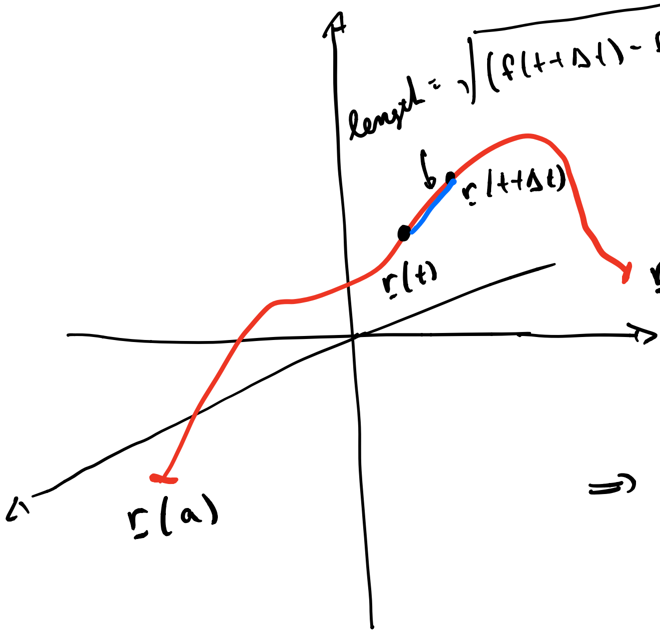
Integration of vector functions. Just integrate componentwise:

$$\int_a^b \langle x(t), y(t), z(t) \rangle \, dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle.$$

§10.8: Arc length and ~~curvature~~.

How can we compute the arc length of a space curve? $(\underline{r}(t),\ a \le t \le b)$

$= \langle f(t), g(t), h(t) \rangle$



$$\text{length} = \sqrt{(f(t+\Delta t)-f(t))^2 + (g(t+\Delta t)-g(t))^2 + (h(t+\Delta t)-h(t))^2}$$

$$\approx \sqrt{(\Delta t \cdot f'(t))^2 + (\Delta t \cdot g'(t))^2 + (\Delta t \cdot h'(t))^2}$$

$$= \Delta t \cdot \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$$

$$= \int_a^b |\underline{r}'(t)|\, dt.$$

$\Longrightarrow$ $\boxed{\text{arc length of curve, from } \underline{r}(a) \text{ to } \underline{r}(b) = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt.}$

(similarly for a plane curve)

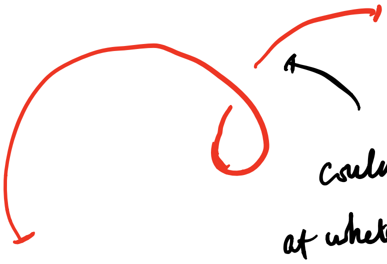
Ex 1. Length of one arc of helix? $\underline{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

$$L = \int_0^{2\pi} |\underline{r}'(t)|\, dt = \int_0^{2\pi} |\langle -\sin t, \cos t, 1 \rangle|\, dt$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\,\pi.$$

An important distinction: $\underline{r}(t)$ is <u>not</u> the same thing as the curve it represents!



could traverse this at whatever speed we like!

E.g., $\underline{r}_1(t) = \langle t, t^2, t^3 \rangle$, $1 \le t \le 2$,

$\underline{r}_2(t) = \langle e^u, e^{2u}, e^{3u} \rangle$, $0 \le u \le \log 2$,

represent the same curve!

<u>Note</u>: $\boxed{\underline{r}_2(t) = \underline{r}_1(e^u)}$.

Two <u>parametrizations</u> of the same curve.

A canonical parametrization. One way to parametrize a curve is by arclength. That is, travel along it at a constant speed of 1.

Given $\underline{r}(t)$, rep'ing C, how to find that constant-speed parametrization?

- first, define the arc length function:

$$\boxed{s(t) := \int_a^t |\underline{r}'(t)|\,dt, \qquad a \le t \le b.} \qquad (*)$$

"length of C from $\underline{r}(a)$ to $\underline{r}(t)$" [DRAW PICTURE]

- now, attempt to solve $(*)$ for t as a function of s. If successful, can write $t = t(s)$.

- the desired "arc length parametrization" is:

$$\boxed{\underline{r}(t(s)), \qquad 0 \le s \le L.}$$

10.8.10. Arc length parametrization of $\underline{r}(t) = \langle e^{2t}\cos 2t,\ 2,\ e^{2t}\sin 2t\rangle$? (from $t=0$)

a. $\underline{r}'(t) = \langle 2e^{2t}\cos 2t - 2e^{2t}\sin 2t,\ 0,\ 2e^{2t}\sin 2t + 2e^{2t}\cos 2t\rangle$

$= 2e^{2t}\langle \cos 2t - \sin 2t, 0, \sin 2t + \cos 2t\rangle$

$\Rightarrow |\underline{r}'(t)| = 2e^{2t}\sqrt{(\cos 2t - \sin 2t)^2 + 0^2 + (\sin 2t + \cos 2t)^2}$

$= 2e^{2t}\sqrt{(\cos^2 2t - \cancel{2\cos 2t\sin 2t} + \sin^2 2t) + (\sin^2 2t + \cancel{2\sin 2t\cos 2t} + \cos^2 2t)}$

$$= 2e^{2t}\sqrt{2\cos^2 2t + 2\sin^2 2t}$$

$$= 2\sqrt{2}\, e^{2t}.$$

$$\Rightarrow \quad s(t) = \int_0^t 2\sqrt{2}\, e^{2t}\, dt = \left[2\sqrt{2}\cdot \tfrac{1}{2} e^{2t}\right]_0^t$$

$$= \sqrt{2}\left(e^{2t}-1\right)$$

$$\Rightarrow \quad s = \sqrt{2}\left(e^{2t}-1\right) \Rightarrow e^{2t}-1 = \frac{\sqrt{2}}{2}s$$

$$\Rightarrow e^{2t} = \frac{\sqrt{2}}{2}s+1$$

$$\Rightarrow 2t = \log\left(\frac{\sqrt{2}}{2}s+1\right)$$

$$\Rightarrow t = \frac{1}{2}\log\left(\frac{\sqrt{2}}{2}s+1\right).$$

Now, $e^{2\cdot\left(\frac{1}{2}\log\left(\frac{\sqrt{2}}{2}s+1\right)\right)} = \frac{\sqrt{2}}{2}s+1.$

$$\Rightarrow \underline{r}\left(t(s)\right) = \left\langle \left(\frac{\sqrt{2}}{2}s+1\right)\cos\left(\log\left(\frac{\sqrt{2}}{2}s+1\right)\right),\ 2,\ \left(\frac{\sqrt{2}}{2}s+1\right)\sin\left(\log\left(\frac{\sqrt{2}}{2}s+1\right)\right)\right\rangle$$

... now onto §11!

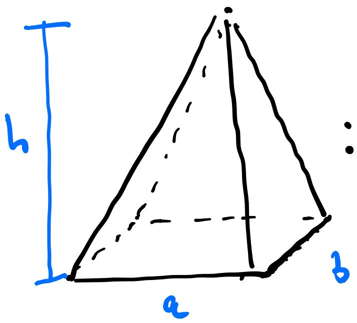
OH today: 1-2:30.

§11: Partial derivatives

Goal: understand analogues of $\frac{df}{dx}$ for functions of multiple variables.

§11.1: Functions of several variables

$$\begin{array}{l} f: \; U \longrightarrow \mathbb{R}. \\ \cap \\ \mathbb{R}^n \end{array}$$

Ex. • volume of 

$:= \boxed{V(a,b,h) = \frac{1}{3}abh}$

• time it takes to shell n beans at a rate of r $\frac{\text{beans}}{\text{sec}}$ $:= \boxed{t(n,r) = \frac{n}{r} \text{ secs}}$

• elevation @ longitude x, latitude y $:= \boxed{E(x,y)}$.

Just as with functions of 1 var, $f(x,y)$ has a

domain, **range**.

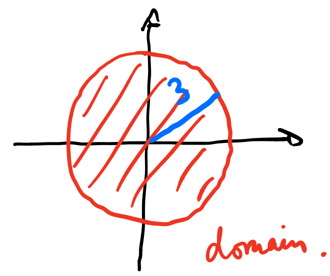
set of (x,y) where $f(x,y)$ is defined.

set of real numbers "hit" by f.

Ex 1. $f(x,y) := \sqrt{9-x^2-y^2}$. Domain, range?

- domain? need $9-x^2-y^2 \geq 0$

  $\iff x^2+y^2 \leq 9.$



(for (x,y) in range)

- range? $0 \leq 9-x^2-y^2 \leq 9$

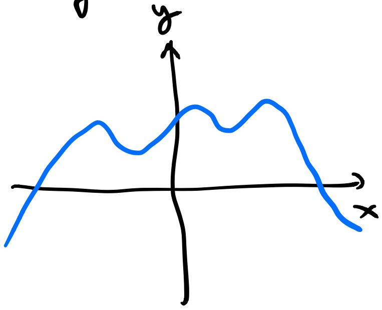
  $\implies 0 \leq \sqrt{9-x^2-y^2} \leq 3$

  $\implies$ range is $[0,3]$.

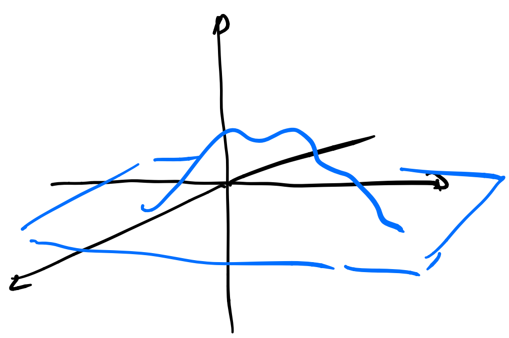
$\triangle$

Visualizing $z = f(x,y)$.

$y = f(x)$



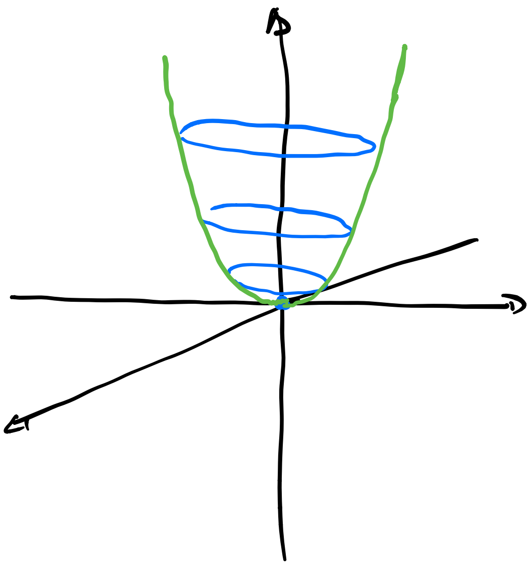
$z = f(x,y)$



slices **Ex 5**: Draw $z = \underbrace{4x^2+y^2}_{f(x,y)}$.

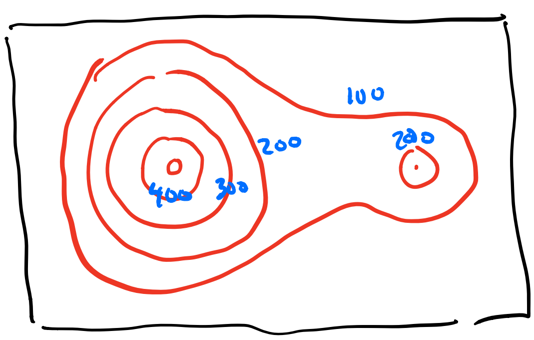
take z-slices: take intersection w/ $\{z=a\}$.

$$z = 4x^2+y^2 \overset{z=a}{\rightsquigarrow} 4x^2+y^2 = a \begin{cases} \text{ellipse w/ ints } \pm\frac{\sqrt{a}}{2}, \pm\sqrt{a}, & a>0, \\ \{(0,0)\}, & a=0 \\ \emptyset, & a<0. \end{cases}$$

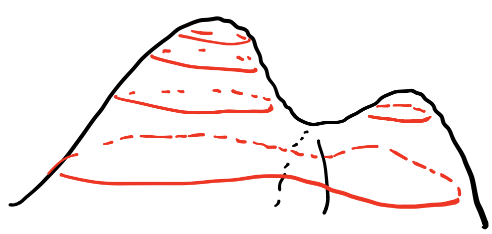


In general, curves $f(x,y)=a$ are called isoclines or contour lines or level curves.

Level curves: draw all contour lines on single copy of xy-plane.



topographic map



Miscellaneous. Sometimes, can just figure it out!

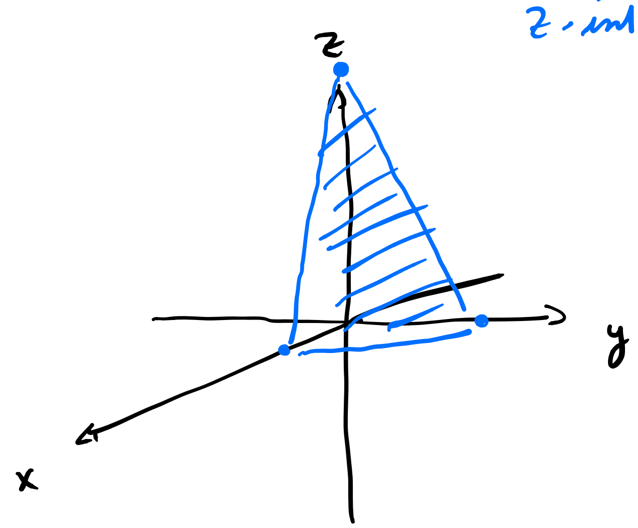
Ex 3.4. • Draw $z = 6 - 3x - 2y$.

$$z = 6 - 3x - 2y \implies 3x + 2y + z = 6$$

x-int? $3x = 6 \implies x = 2$

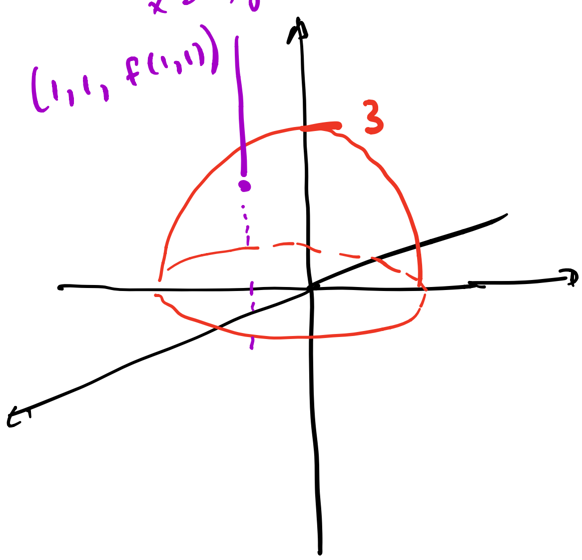
y-int? $2y = 6 \implies y = 3$

z-int? $z = 6$.



• $z = \sqrt{9 - x^2 - y^2} \implies z^2 = 9 - x^2 - y^2$

$\implies x^2 + y^2 + z^2 = 9$.



§11.1: Functions of several variables (con't).

We've seen <u>level curves</u> $f(x,y) = a$ ... how about for a function of 3 variables? $f(x,y,z) = a$ is a <u>level surface</u>.

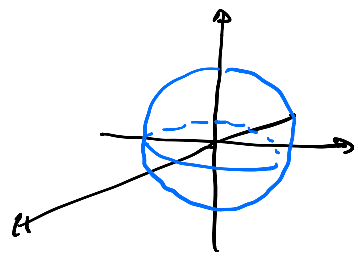
<u>Ex</u>. the temperature @ (x,y,z) is $T(x,y,z) := 200 - x^2 - y^2 - z^2$.

Where is the temp 100?

<u>A</u>. $T(x,y,z) = 100 \iff 200 - x^2 - y^2 - z^2 = 100$

$\iff x^2 + y^2 + z^2 = 100$

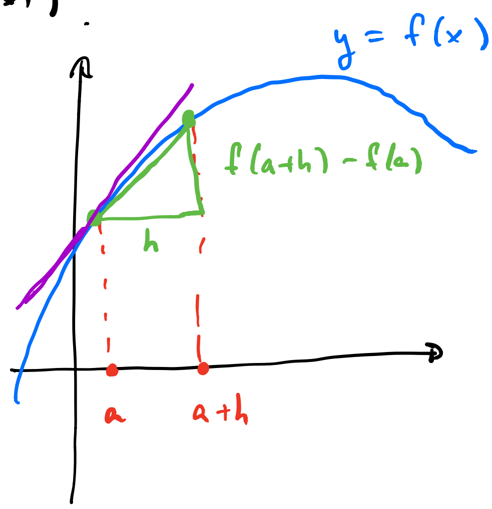
(sphere, radius 10)



§11.3: Partial derivatives.

Recall the defn of $\frac{df}{dx}$ (for $f = f(x)$).

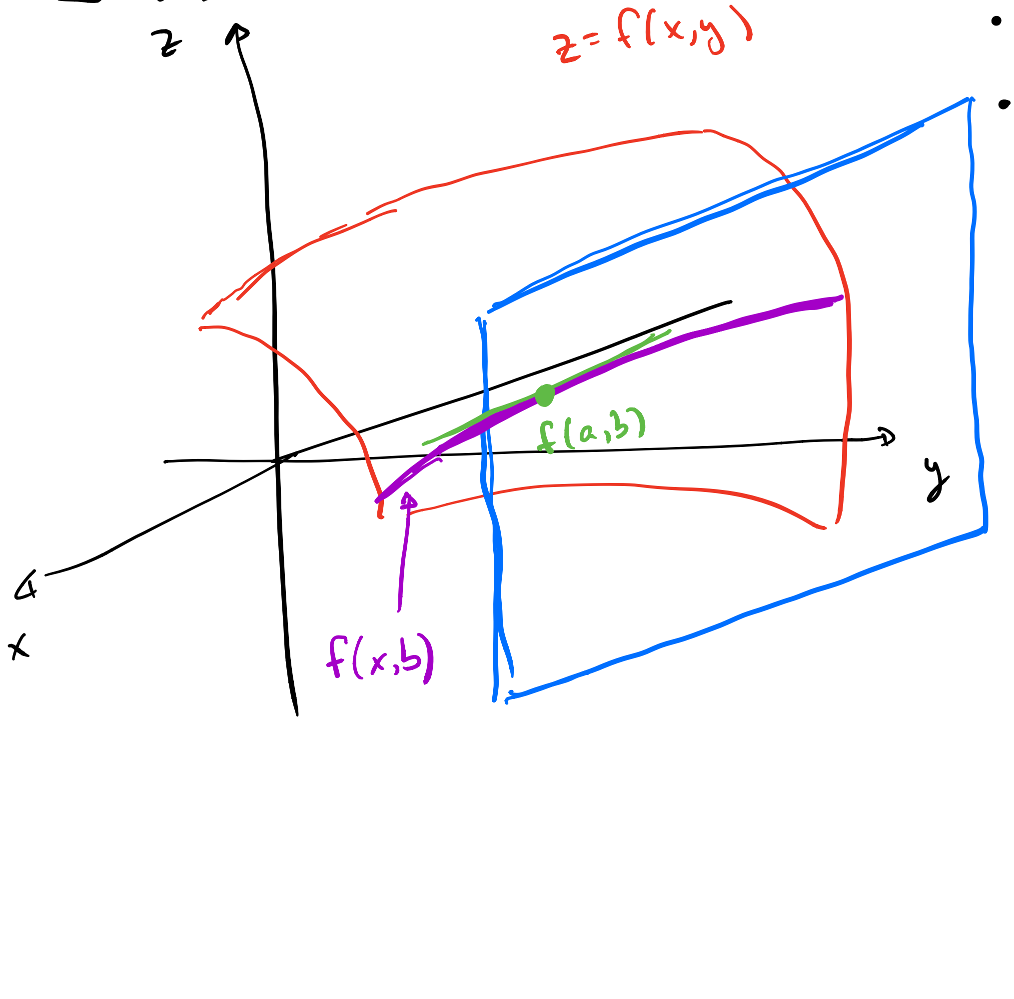
$$\boxed{\frac{df}{dx}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$



We can do something similar for $f(x,y)$!

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x}(a,b) &:= f_x(a,b) := \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} \\ \frac{\partial f}{\partial y}(a,b) &:= f_y(a,b) := \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h} \end{aligned}}$$

<u>a geometric interpretation of $\frac{\partial f}{\partial x}$.</u>



To compute $\frac{\partial f}{\partial x}(a,b)$, we:

- restrict $f(x,y)$ to $y=b$
- taking the slope of purple curve @ $x=a$.

So: $\boxed{\text{to compute } \frac{\partial f}{\partial x}, \text{ think of } y \text{ as a constant, and we differentiate, thinking of } x \text{ as the variable.}}$

Ex 1. $f(x,y) := x^3 + x^2y^3 - 2y^2$. f_x, f_y? $f_x(2,1), f_y(2,1)$?

a. $f_x = \boxed{3x^2 + 2xy^3}$, $f_y = \boxed{3x^2y^2 - 4y}$.

$f_x(2,1) = 3\cdot 4 + 2\cdot 2\cdot 1$ $\boxed{= 16}$.

$f_y(2,1) = 3\cdot 4\cdot 1 - 4\cdot 1$ $\boxed{= 8}$.

Ex 3. $f(x,y) := \sin\left(\frac{x}{1+y}\right)$. f_x, f_y?

- $f_x = \cos\left(\frac{x}{1+y}\right)\cdot\frac{1}{1+y} = \boxed{\frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)}$.
- $f_y = \cos\left(\frac{x}{1+y}\right)\cdot\left(-\frac{x}{(1+y)^2}\right) = \boxed{-\frac{x}{(1+y)^2}\cdot\cos\left(\frac{x}{1+y}\right)}$.

Δ

Can form higher partial derivatives just by differentiating more than once. eg.: $(f_x)_y =: f_{xy}$.

$f_{xyyyxxyx}$.

Ex. Compute f_{xy}, f_{yx} in the case of $f = \sin\left(\frac{x}{1+y}\right)$.

A. • $f_{xy} = (f_x)_y$

$$= \left(\frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)\right)_y$$

$$= \left(\frac{-1}{(1+y)^2}\right)\cos\left(\frac{x}{1+y}\right) + \frac{1}{1+y}\left(+\sin\left(\frac{x}{1+y}\right)\cdot\left(\frac{\cancel{-}x}{(1+y)^2}\right)\right)$$

$$= \boxed{\frac{-1}{(1+y)^2}\cos\left(\frac{x}{1+y}\right) + \frac{x}{(1+y)^3}\sin\left(\frac{x}{1+y}\right).}$$

• $f_{yx} = (f_y)_x$

$$= \left(-\frac{x}{(1+y)^2}\cdot\cos\left(\frac{x}{1+y}\right)\right)_x$$

$$= -\frac{1}{(1+y)^2}\cos\left(\frac{x}{1+y}\right) - \frac{x}{(1+y)^2}\cdot\left(-\sin\left(\frac{x}{1+y}\right)\cdot\frac{1}{1+y}\right)$$

$$= \boxed{-\frac{1}{(1+y)^2}\cos\left(\frac{x}{1+y}\right) + \frac{x}{(1+y)^3}\sin\left(\frac{x}{1+y}\right).}$$

Δ

Thm (Clairaut): Under "reasonable smoothness hypotheses",

$$f_{yx} = f_{xy}$$

(in general: order doesn't matter.)

Ex 4. Suppose z is defined implicitly by:

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

A Differentiate implicitly:

$$x^3 + y^3 + z^3 + 6xyz = 1 \implies \frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 + 6xyz\right) = 0$$

$$\implies 3x^2 + 3z^2 \cdot \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\implies \left(3z^2 + 6xy\right)\frac{\partial z}{\partial x} + \left(3x^2 + 6yz\right) = 0$$

$$\implies \boxed{\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}}.$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}}.$$

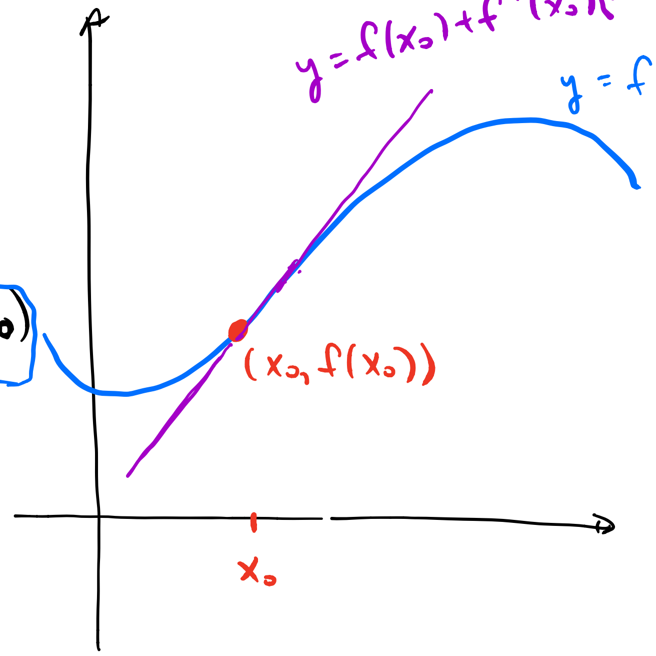
Δ

§11.4: Tangent planes, linear approximations

Recall the tangent line to $y = f(x)$ @ $x = x_0$.

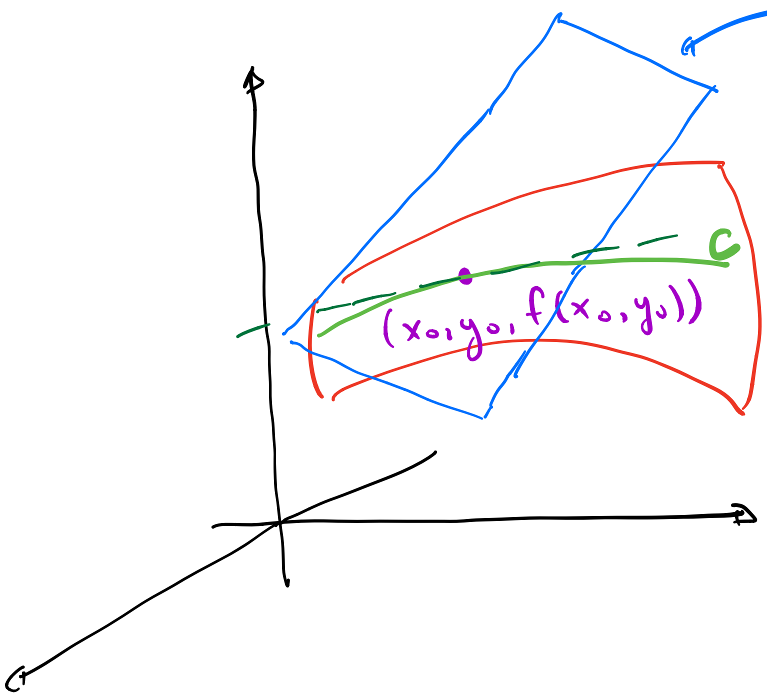
Defined by $\boxed{y = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{}}$

linear approximation to f near $x = x_0$.



Today will do the analogue, for $f = f(x,y)$.

tangent plane P to $z = f(x,y)$ @ $(x,y) = (x_0, y_0)$.



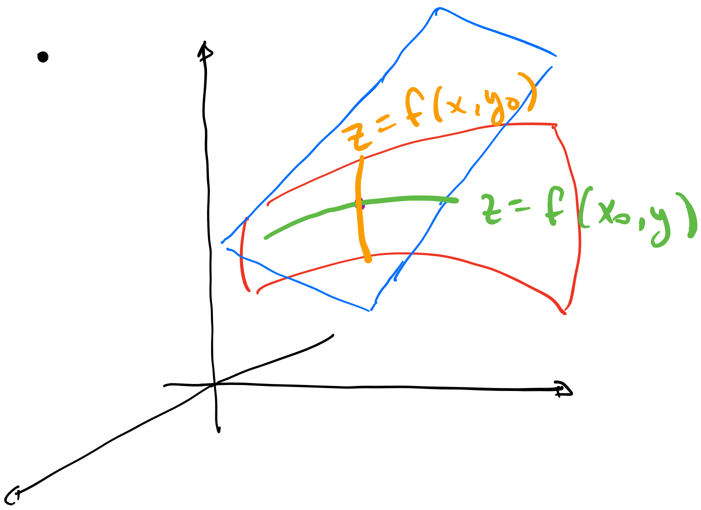
note: for every curve C on S thru $(x_0, y_0, f(x_0, y_0))$, the tangent line to C lies in P.

Lets derive a formula for P.

- (almost) any plane can be written as $z - z_0 = a(x - x_0) + b(y - y_0)$.

(why: start w/ $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$; divide by C, manipulate)

$$P: \quad z - f(x_0, y_0) = a(x - x_0) + b(y - y_0). \qquad (*)$$

- 

P contains tangent lines to these 2 curves, i.e.:

$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$

$z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$.

- $(*) \overset{y = y_0}{\rightsquigarrow} \quad z - f(x_0, y_0) = a(x - x_0) \rightsquigarrow a := f_x(x_0, y_0)$

  $(*) \overset{x = x_0}{\rightsquigarrow} \quad z - f(x_0, y_0) = b(y - y_0) \rightsquigarrow b = f_y(x_0, y_0)$.

$\Rightarrow$ tangent plane to $z = f(x, y)$ @ $(x_0, y_0, f(x_0, y_0))$

is $$\boxed{z = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x,y)\text{, the linear approximation to } f(x,y) \text{ @ } (x_0, y_0).}}$$

Ex 2. Find the linear approximation to $f(x,y) = xe^{xy}$ @ $(1,0)$. Use this to approximate $f(1.1, -0.1)$.

A. $f_x = e^{xy} + x \cdot (ye^{xy})$ $\Rightarrow$ $f_x(1,0) = e^0 + 0 \cdot e^0$
$= e^{xy} + xye^{xy}$ $= 1.$

$f_y = x^2 e^{xy} \Rightarrow f_y(1,0) = 1 \cdot e^0 = 1.$

$$L(x,y) = \underbrace{f(1,0)}_{1} + \underbrace{f_x(1,0)}_{1} \cdot (x-1) + \underbrace{f_y(1,0)}_{1} \cdot (y-0)$$

$$= 1 + (x-1) + (y-0) = x+y.$$

$$f(1.1, -0.1) \approx L(1.1, -0.1) = 1.1 - 0.1 = 1.$$

Differentials.

$$\underbrace{z - f(x_0,y_0)}_{\Delta z} = f_x(x_0,y_0)\underbrace{(x-x_0)}_{\Delta x} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{\Delta y}$$

$$\Rightarrow \boxed{\Delta z \approx f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y.}$$

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy.$$ "differential"

Spring '12, #1. $f := x^3 + x(y^3+1) + e^{y-1}$.

(a) $f(2,1)$

(b) $f_x(2,1)$, $f_y(2,1)$?

(c) Estimate change in f if x increased by 0.2, y increased by 0.3.

(d) If we increase x by 0.2, how much to change y by so f remains $\approx$ the same?

A. (a) $f(2,1) = 2^3 + 2\cdot(1+1) + e^0 = 13.$

(b) $f_x = 3x^2 + (y^3+1) \to f_x(2,1) = 3\cdot 4 + (1+1) = 14.$

$f_y = x\cdot(3y^2) + e^{y-1} \to f_y(2,1) = 2\cdot 3 + 1 = 7.$

$$\begin{aligned} f(x,y) \approx L(x,y) &= f(2,1) + f_x(2,1)\cdot(x-2) + f_y(2,1)(y-1) \\ &= 13 + 14(x-2) + 7(y-1) \\ &= 13 + 14x - 28 + 7y - 7 \\ &= 14x + 7y - 22. \end{aligned}$$

(c) $$\begin{aligned} \Delta f &\approx f_x(2,1)\,\Delta x + f_y(2,1)\,\Delta y \\ &= 14\cdot 0.2 + 7\cdot 0.3 = \frac{14}{5} + 7\cdot\frac{3}{10} = \frac{49}{10}. \end{aligned}$$

(d) $$\overset{0}{\Delta f} \approx f_x(2,1)\,\overset{0.2}{\Delta x} + f_y(2,1)\,\Delta y$$

$$\Rightarrow \quad 0 = 14 \cdot 0.2 + 7\,\Delta y$$

$$\Rightarrow \quad \Delta y = \frac{-14 \cdot 0.2}{7} = -0.4.$$

OH today: 1 - 2:30.

Practice midterm 1 posted!

Kaylee's review: Tu, 4-6 PT.

§11.5: The chain rule.

Recall **THE GLORIOUS CHAIN RULE:**

$$\boxed{\frac{d}{dt}\left[f(x(t))\right] = \frac{df}{dx} \cdot \frac{dx}{dt} = \frac{df}{dx}(x(t)) \cdot \frac{dx}{dt}(t).}$$

We have several versions of CR for functions of > 1 variables.

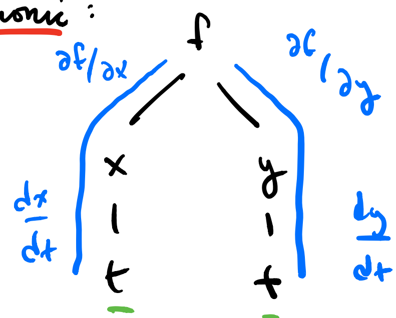
<u>The chain rule, case 1.</u> Say $f = f(x,y)$, $x = x(t)$, $y = y(t)$.

So, $f = f(x(t), y(t))$.

$$\frac{d}{dt}\left[f(x(t), y(t))\right](t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}(x(t), y(t))\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\,\frac{dy}{dt}(t).$$

Mnemonic:



Ex 1. Say $z = x^2y + 3xy^4$, $\quad x = \sin 2t$, $\quad y = \cos t$.

$\qquad\qquad\qquad\qquad\qquad\qquad x(0) = 0 \qquad\qquad y(0) = 1$

Find $\dfrac{dz}{dt}$ @ $t = 0$.

a. $$\frac{dz}{dt} \overset{(*)}{=} \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\Rightarrow \frac{dz}{dt}(0) = \frac{\partial z}{\partial x}(\overset{x(0)}{0}, \overset{y(0)}{1})\,\frac{dx}{dt}(\overset{t=0}{0}) + \frac{\partial z}{\partial y}(0,1)\,\frac{dy}{dt}(\overset{t=0}{0}).$$

- $\dfrac{dx}{dt} = 2\cos 2t \Rightarrow \dfrac{dx}{dt}(0) = 2$; $\qquad \dfrac{dy}{dt} = -\sin t \Rightarrow \dfrac{dy}{dt}(0) = 0.$

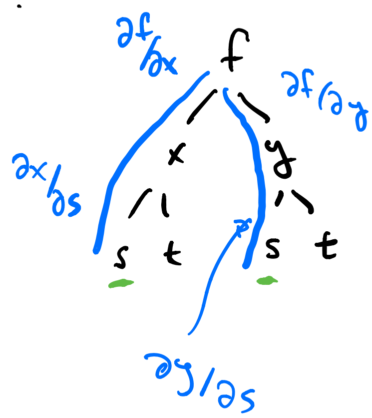
- $\dfrac{\partial z}{\partial x} = 2xy + 3y^4 \Rightarrow \dfrac{\partial z}{\partial x}(0,1) = 0 + 3 = 3.$

  $\dfrac{\partial z}{\partial y} = x^2 + 12xy^3 \Rightarrow \dfrac{\partial z}{\partial y}(0,1) = 0 + 0 = 0,$

$$(*) \Rightarrow \frac{dz}{dt}(0) = 3\cdot 2 + 0\cdot 0 = 6. \qquad \triangle$$

The chain rule, case 2. $\qquad f = f(x(s,t), y(s,t))$.

$$\frac{\partial}{\partial s}\Big(f(x(s,t), y(s,t))\Big) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$



$$\frac{\partial}{\partial t}\Big(f(x(s,t), y(s,t))\Big) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Fall '15, # 1. Say $z = f(x,y)$, where:

- $f(1,3) = 7$,
- $f_x(1,3) = 5$,
- $f_y(1,3) = -4$.

Say $x = st$, $y = 2s + t$.

(a) Find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$ @ $(s,t) = (1,1)$.

(b) Approximate z @ $(s,t) = (0.9, 1.1)$.

(a) $z = z(x(s,t), y(s,t)) \Rightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$.

- $x(1,1) = 1$, $y(1,1) = 3$.

$$\Rightarrow \frac{\partial z}{\partial s}(1,1) = \frac{\partial z}{\partial x}(\underbrace{x(1,1)}_{1}, \underbrace{y(1,1)}_{3})\frac{\partial x}{\partial s}(1,1) + \frac{\partial z}{\partial y}(\overbrace{x(1,1)}^{1}, \overbrace{y(1,1)}^{3}) \cdot \frac{\partial y}{\partial s}(1,1)$$

$$= \underbrace{\frac{\partial z}{\partial x}(1,3)}_{5}\underbrace{\frac{\partial x}{\partial s}(1,1)}_{1} + \underbrace{\frac{\partial z}{\partial y}(1,3)}_{-4}\underbrace{\frac{\partial y}{\partial s}(1,1)}_{2}.$$

- $\frac{\partial x}{\partial s} = t$, $\frac{\partial y}{\partial s} = 2$, $\frac{\partial x}{\partial t} = s$, $\frac{\partial y}{\partial t} = 1$.

$$\Rightarrow \frac{\partial z}{\partial s}(1,1) = 5 \cdot 1 + (-4) \cdot 2 = 5 - 8 = -3.$$

Similarly, $$\frac{\partial z}{\partial t}(1,1) = \underbrace{\frac{\partial z}{\partial x}(1,3)}_{5}\underbrace{\frac{\partial x}{\partial t}(1,1)}_{1} + \underbrace{\frac{\partial z}{\partial y}(1,3)}_{-4}\underbrace{\frac{\partial y}{\partial t}(1,1)}_{1}$$

$$= 5 \cdot 1 + (-4) \cdot 1 = 1$$

(b) Approximate z @ $(s,t) = (0.9, 1.1)$.

$L(s,t) :=$ linear approx. to z @ $(1,1)$

$$\Rightarrow \quad L(s,t) = \underset{7}{z(\overset{s}{1},\overset{t}{1})} + \underset{-3}{\frac{\partial z}{\partial s}(1,1)}(s-1) + \underset{1}{\frac{\partial z}{\partial t}(1,1)}(t-1)$$

$$\Rightarrow \quad z(s=0.9,\ t=1.1) \approx L(0.9, 1.1)$$

$$= 7 - 3\cdot(-0.1) + (0.1)$$

$$= 7 + 0.3 + 0.1 = 7.4.$$

Δ

<u>Spring '12, #2.</u> A spaceship flies thru space. Temp is $T(x,y,z) := e^{xy+z}$, path is $\underline{r}(t) := \langle t, t^2, t^3 \rangle$.

(a) Velocity @ $t=2$? $\underline{r}'(2)$ $\quad \sqrt{161}$

(b) Rate of change of temp @ $t=2$? $\quad \frac{d}{dt}\left[T(t,t^2,t^3)\right](2).$

$24e^{16}.$

§11.6: Directional derivatives, gradient vector.

Recall,

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} \\ \frac{\partial f}{\partial y} &= \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}. \end{aligned}}$$

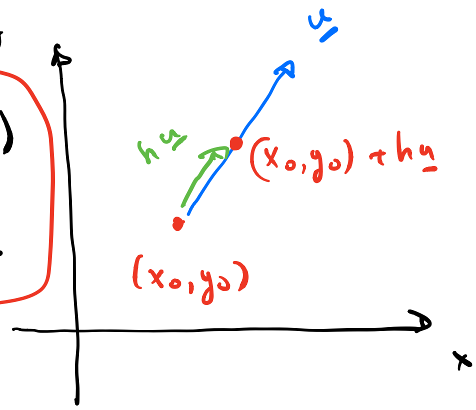
"derivative in direction $\underline{i}$"

"derivative in direction $\underline{j}$"

What if we want to differentiate in some other direction?

Fix $\underline{u} = \langle a, b \rangle$, $|\underline{u}| = 1$.

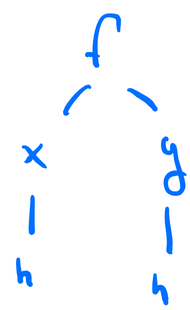
$$\boxed{D_{\underline{u}} f := \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0, y_0)}{h}.}$$



Note, $D_{\underline{i}} f = \dfrac{\partial f}{\partial x}$, $D_{\underline{j}} f = \dfrac{\partial f}{\partial y}$.

A convenient formula for $D_{\underline{u}} f$.

Note, $D_{\underline{u}} f = \dfrac{d}{dh}\Big|_{h=0} \left[f(x_0 + ha, y_0 + hb) \right]$



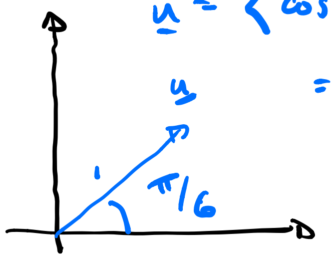
$$\overset{CR}{=} \frac{\partial f}{\partial x} \cdot \underbrace{\frac{d}{dh}(x_0 + ha)}_{a} + \frac{\partial f}{\partial y} \cdot \underbrace{\frac{d}{dh}(y_0 + hb)}_{b}$$

$(\underline{u} = \langle a, b \rangle)$

$$= a \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y}.$$

$$\Longrightarrow \boxed{D_{\underline{u}} f = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}.}$$

Ex 1. $f := x^3 - 3xy + 4y^2$, $\underline{u} :=$ unit vector w/ angle $\pi/6$.



$$\underline{u} = \langle \cos \pi/6, \sin \pi/6 \rangle = \langle \sqrt{3}/2, 1/2 \rangle.$$

- $\dfrac{\partial f}{\partial x} = 3x^2 - 3y$
- $\dfrac{\partial f}{\partial y} = -3x + 8y$.

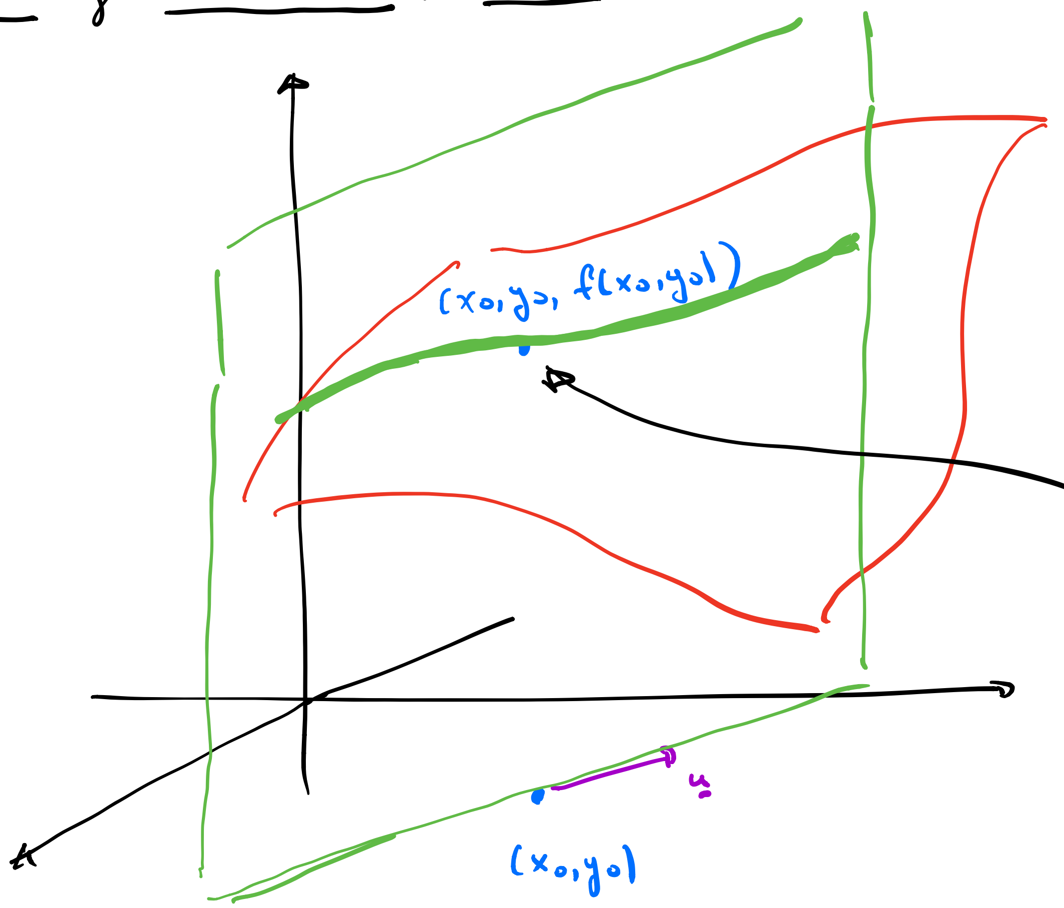
$$\Rightarrow D_{\underline{u}} f = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}$$

$$= \frac{\sqrt{3}}{2} \cdot (3x^2 - 3y) + \frac{1}{2} \cdot (-3x + 8y)$$

$$= \boxed{\frac{3\sqrt{3}}{2} x^2 + \left(4 - \frac{3\sqrt{3}}{2}\right) y - \frac{3}{2} x.}$$

$\triangle$

The geometric interpretation.



$z = f(x,y)$

$P :=$ plane containing $\underline{u}$, $\underline{k}$.

$D_{\underline{u}} f(x_0, y_0)$ is the slope of the green curve @ $(x_0, y_0, f(x_0, y_0))$.

The gradient vector.

Note, $$D_{\underline{u}} f = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}$$

$\langle a, b \rangle$

$$= \langle a, b \rangle \cdot \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}_{=:\ \nabla f,\ \text{the gradient of } f.}.$$

$$\Rightarrow \boxed{D_{\underline{u}} f = \underline{u} \cdot \nabla f.}$$

Note, $$D_{\underline{u}} f = \underline{u} \cdot \nabla f = |\underline{u}| |\nabla f| \cos\theta$$
$$= |\nabla f| \cos\theta$$

$\Rightarrow$ $D_{\underline{u}} f$ is maximized when $\underline{u}$, ∇f are pointing in the same direction

$\Rightarrow$ $\boxed{\text{$\nabla f$ is pointing in the direction that f is increasing the fastest, and this rate is $|\nabla f|$.}}$

Fall '13, 2c. $f := x^2 + xy + y^2$. @ $(1,2)$, in which direction is f increasing the fastest?

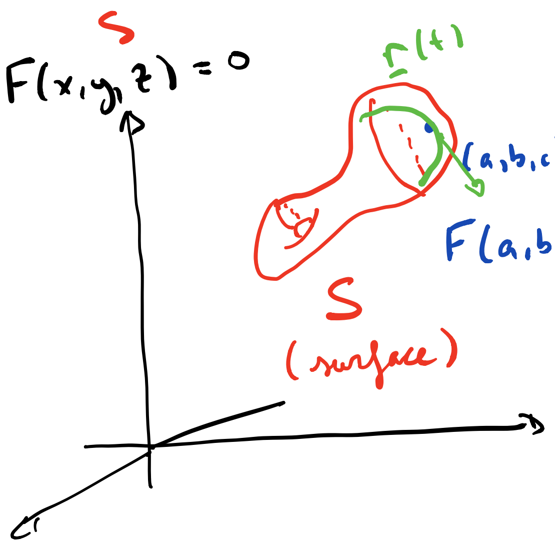
a. $$\frac{\nabla f(1,2)}{|\nabla f(1,2)|}.$$

$$\nabla f = \langle 2x+y,\ x+2y \rangle \Rightarrow \nabla f(1,2) = \langle 4, 5 \rangle.$$

$$\Rightarrow \quad \underline{u} = \frac{\langle 4,5\rangle}{|\langle 4,5\rangle|} = \boxed{\left\langle \frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}} \right\rangle}. \qquad \triangle$$

$\sqrt{16+25} = \sqrt{41}$

Tangent planes to level surfaces.



What's the tangent plane P to S @ (a,b,c)?

Say $\underline{r}(t)$ is any curve in S, w/ $\underline{r}(0) = \langle a,b,c\rangle$.

$= \langle x(t), y(t), z(t)\rangle$

$\Rightarrow$ $\underline{r}'(0)$ is a tangent vector to S @ (a,b,c)

$$F(x(t), y(t), z(t)) = 0$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} \Big(F(x(t), y(t), z(t)) \Big) = 0$$

$$\overset{CR}{\Rightarrow} \quad \frac{\partial F}{\partial x}(\underline{r}(0))\, \frac{dx}{dt}(0) + \frac{\partial F}{\partial y}(\underline{r}(0))\, \frac{dy}{dt}(0) + \frac{\partial F}{\partial z}(\underline{r}(0))\, \frac{dz}{dt}(0) = 0$$

$$\Rightarrow \nabla F(\underline{r}(0)) \cdot \underline{r}'(0) = 0$$

$$\Rightarrow \underline{r}'(0) \perp \nabla F(\underline{r}(0))$$

$\Longrightarrow$ $\boxed{\nabla F(a,b,c) \text{ is perp. to } S \text{ @ } (a,b,c).}$

$\Rightarrow$ Tangent plane to S @ $\underbrace{(a,b,c)}^{= p_0}$:

$$\boxed{\nabla F(p_0) \cdot (p - p_0) = 0.}$$

Midterm coverage: §§ 10.1 – 10.8, 11.1, 11.3 – 11.6.

no curvature

§11.6: Directional derivatives, the gradient vector.

Here are the marquee properties of the gradient:

(0) For any function $F : \mathbb{R}^n \to \mathbb{R}$, can form its gradient

$$\nabla F(x_1, \dots, x_n) := \left\langle \frac{\partial F}{\partial x_1}(x_1, \dots, x_n), \dots, \frac{\partial F}{\partial x_n}(x_1, \dots, x_n) \right\rangle.$$

Eg, $f = f(x,y) \rightsquigarrow \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$,

$f = f(x,y,z) \rightsquigarrow \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$.

(1) If $\underline{u}$ is a unit vector,

$$D_{\underline{u}} F = \nabla F \cdot \underline{u}.$$

(2) ∇F is pointing in the direction that F is increasing most rapidly. This maximal rate of change is $|\nabla F|$.

or $F = \lambda$, λ a constant

(3) If S is the surface $S := \{F(x,y,z) = 0\}$, then @ the point (a,b,c) on S, $\nabla F(a,b,c)$ is perpendicular to S.

(4) $$\frac{d}{dt}\left[f(\underline{r}(t))\right] = \nabla f \cdot \underline{r}'(t).$$

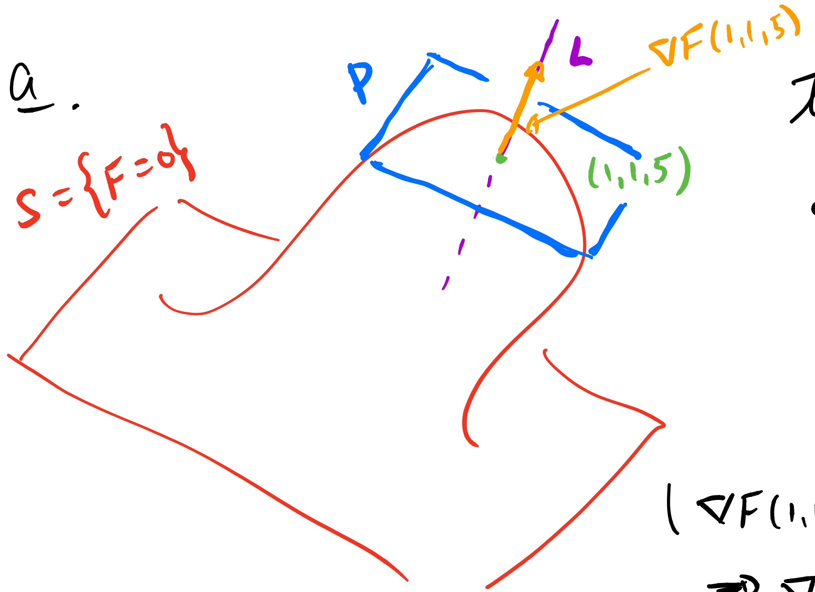
Fall '11, #7. $\qquad S := \{\underbrace{x^2+xy+3y^2-z}_{F}=0\}$.

(a) Parametric equation for line thru $(1,1,5)$, perp. to S.

(b) Tangent plane to S @ $(1,1,5)$?

(c) Rate of change of F in the direction $\underline{v} := \langle 1,2,3\rangle$.

a.



To answer (a), (b), need ...

- point on L, vector parallel to L
  $(1,1,5)$ $\qquad \nabla F(1,1,5)$
- point on P, vector perp. to P.
  $(1,1,5)$ $\qquad \nabla F(1,1,5)$

($\nabla F(1,1,5)$ is perp. to S @ $(1,1,5)$
$\Rightarrow \nabla F(1,1,5)$ also perp. to P @ $(1,1,5)$)

Let's check that $(1,1,5)$ is on S: $F(1,1,5) = 1+1+3-5=0$. ✓

$$\nabla F = \nabla(x^2+xy+3y^2-z) = \langle 2x+y,\ x+6y,\ -1\rangle \Rightarrow \nabla F(1,1,5) = \langle 3,7,-1\rangle.$$

(a) Line thru $(1,1,5)$, parallel to $\langle 3,7,-1\rangle$:

$$\underline{r} = \langle 1,1,5\rangle + t\langle 3,7,-1\rangle$$

$$\Leftrightarrow \boxed{x = 3t+1,\quad y = 7t+1,\quad z = -t+5.}$$

(b) Plane thru $(1,1,5)$, perp. to $\langle 3,7,-1\rangle$:

$$3\cdot(\underbrace{x-1}_{-3}) + 7\cdot(\underbrace{y-1}_{-7}) - 1\cdot(\underbrace{z-5}_{+5}) = 0$$

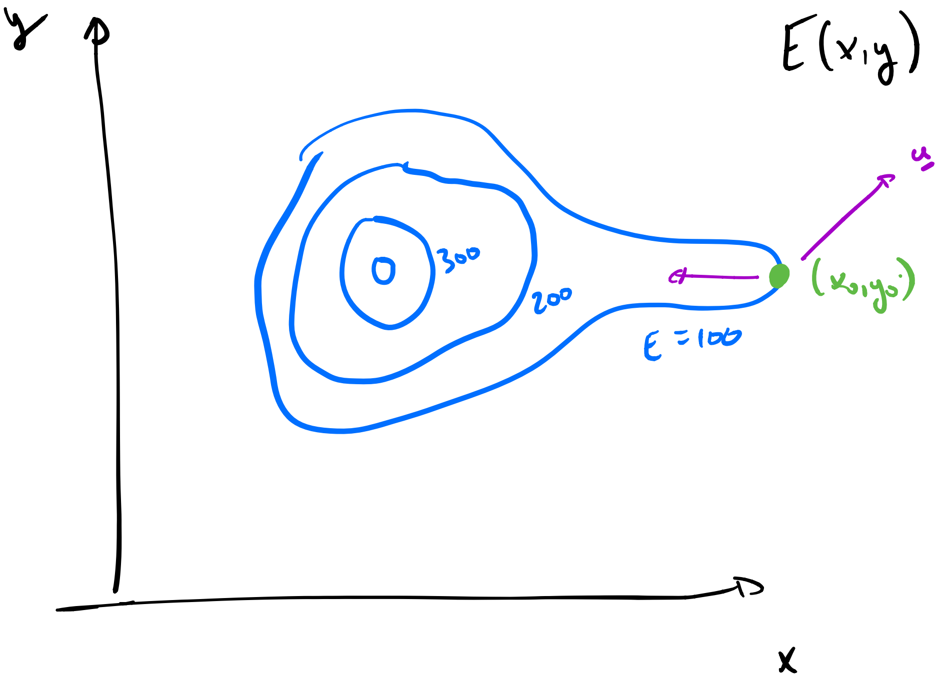
$$\Leftrightarrow \boxed{3x+7y-z=5}$$

(c) Rate of change of F in the direction $\underline{v} := \langle 1,2,3 \rangle$.

$$\underline{u} := \frac{\underline{v}}{|\underline{v}|} = \frac{\langle 1,2,3\rangle}{|\langle 1,2,3\rangle|} = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle.$$

$$D_{\underline{u}} F(1,1,5) = \underbrace{\nabla F(1,1,5)}_{\langle 3,7,-1\rangle} \cdot \underbrace{\underline{u}}_{\frac{1}{\sqrt{14}}\langle 1,2,3\rangle}$$

$$= \frac{14}{\sqrt{14}} = \sqrt{14}.$$



OH today: noon - 1:30.

OH today: 1 - 2:30; special Th OH: 3:30 - 6:30 PT

(default: 226 has priority for 3:30 - 5)

Practice Midterm 1 for MATH 226, section 39559

You have 50 minutes. You may use one, one-sided sheet of notes. You may not use any calculator, cell phone, or similar device.

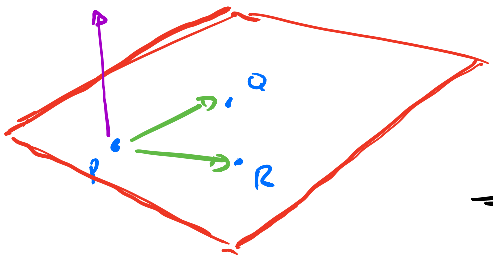
Name: **Date:**

Problem	Score
#1	/10
#2	/10
#3	/10
#4	/10
#5	/10
Total	/50

need pt, normal vector

$\vec{PQ}\times\vec{PR}$

Problem 1: (a) Find an equation for the plane passing through the points $P = (3, 2, 2)$, $Q = (5, -1, 1)$, and $R = (-1, 0, -4)$.



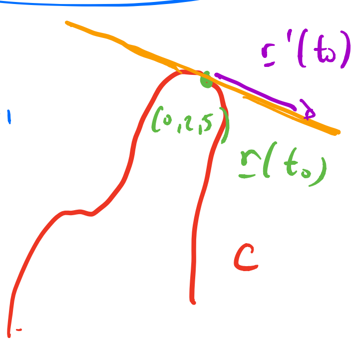
$$\vec{PQ} = \langle 2, -3, -1\rangle$$
$$\vec{PR} = \langle -4, -2, -6\rangle$$
$$\Rightarrow \vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ -4 & -2 & -6 \end{vmatrix} = \langle (-3)(-6)-(-1)(-2),\ -(2(-6)-(-1)(-4)),\ 2(-2)-(-3)(-4)\rangle$$
$$= \langle 16, 16, -16\rangle.$$

$(*)$ $\quad -48 \quad -32 \quad +32$

$$\rightsquigarrow 16(x-3)+16(y-2)-16(z-2)=0$$

(b) Define C to be the curve C parametrized by $\mathbf{r}(t) = \langle 1-t^2, t+1, 2t^2+t+2\rangle$. Find an equation for the line which is tangent to C at $(0, 2, 5)$.

need pt on line, vector parallel to line.



$$r(t_0) = \langle 0,2,5\rangle \Rightarrow t_0+1 = 2 \Rightarrow t_0 = 1$$
$$r'(t) = \langle -2t, 1, 4t+1\rangle \Rightarrow r'(1) = \langle -2, 1, 5\rangle.$$

$$\rightsquigarrow p(t) = \langle 0,2,5\rangle + t\langle -2, 1, 5\rangle, \quad t\in\mathbb{R}.$$
$$\| $$
$$\langle x(t), y(t), z(t)\rangle.$$

(c) Find the point where the line in (b) intersects the plane in (a).

$$(*) \Leftrightarrow 16x+16y-16z = 48 \Leftrightarrow x+y-z=3.$$

$$\rightsquigarrow x(t_1)+y(t_1)-z(t_1) = 3$$
$$\Rightarrow -2t_1 + (t_1+2) - (5t_1+5) = 3$$
$$\Rightarrow -6t_1 - 3 = 3 \Rightarrow -6t_1 = 6$$
$$\Rightarrow t_1 = -1.$$

$$p = \langle 0,2,5\rangle - 1\langle -2,-1,5\rangle = \langle 2, 3, 0\rangle.$$

Problem 2: Consider the curve

$$\mathbf{r}(t) = \left\langle e^t, \frac{\sqrt{2}}{2}e^{2t}, \frac{1}{3}e^{3t} \right\rangle.$$

(a) Compute the length of $\mathbf{r}(t)$, for $0 \leq t \leq 3$.

(b) Suppose that $\rho(s)$ is the reparametrization by arclength of the curve $\mathbf{r}(t)$. Find the length of $\rho(s)$ for $0 \leq s \leq 7$. (Hint: you should not need to do any complicated calculations.)

Problem 3: (a) Define f by

$$f(x,y) = \arctan\left(\log\left(\sqrt{x} + \frac{\cos x}{x^x}\right) - \pi^{1/x}\right) - x^2 y.$$

Compute the partial derivative f_{xxy}. *(Hint: There is a reason that you are only given 1.5".)*

(b) Suppose $u = x^2y^3 + z^4$, where $x = p + 3p^2$, $y = pe^p$, and $z = p \sin p$. Use the chain rule to find u_p.

Problem 4: Let S be the surface in $\mathbb{R}^3$ defined by the equation

$$xz^2 - \arctan(yz) = -\frac{\pi}{4}.$$

$\Rightarrow$ $x \cdot z(x,y)^2 - \arctan\left(y\, z(x,y)\right) = -\frac{\pi}{4}.$

(a) Find expressions for $\partial z/\partial x$ and $\partial z/\partial y$. (Recall that $(\arctan u)' = 1/(1+u^2)$.)

$$z^2 + x \cdot 2z \cdot \frac{\partial z}{\partial x} - \frac{1}{1+y^2z^2} \cdot y \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \left(2xz - \frac{y}{1+y^2z^2}\right)\frac{\partial z}{\partial x} + z^2 = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = - \frac{z^2}{2xz - \frac{y}{1+y^2z^2}}$$

(b) Determine whether $(0,1,1)$ lies on S. Using linear approximation, find an approximation of the z-coordinate of the point on S that has $x = -0.1$ and $y = 1.1$.

(c) Consider a path $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ lying on S that has $\mathbf{r}(0) = \langle 0, 1, 1 \rangle$. Assume that $\frac{dx}{dt}(0) = -2$ and $\frac{dy}{dt}(0) = 1$. Find the value of $\frac{dz}{dt}(0)$.

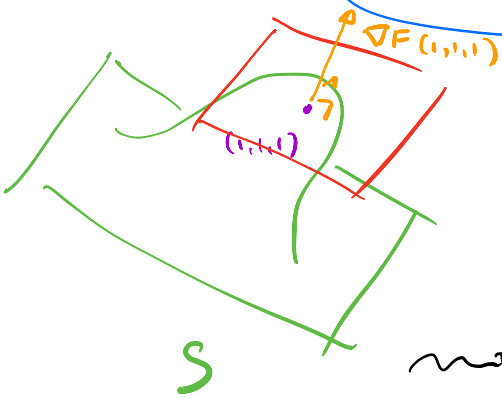
$x^2+y^2-z^2-1=0$

$\Updownarrow$ $\underbrace{x^2+y^2-z^2-1}_{F(x,y,z)}$

need pt, normal vector.

Problem 5: Let S be the hyperboloid defined by $x^2 + y^2 - z^2 = 1$.

(a) Find an equation of the tangent plane P to S at $(1, 1, 1)$.



$\nabla F = \langle 2x, 2y, -2z\rangle$

$\Rightarrow \nabla F(1,1,1) = \langle 2,2,-2\rangle.$

$\rightsquigarrow 2(x-1) + 2(y-1) - 2(z-1) = 0$

$-2 \quad -2 \quad +2$

$\Rightarrow 2x + 2y - 2z = 2$

$\Rightarrow x + y - z = 1.$

$n_2 = \langle 2,2,-2\rangle$

(b) Find all points p on S such that the tangent plane to S at p is parallel to P.

$\langle a,b,c\rangle$

$n_1 = \langle 2a, 2b, -2c\rangle$

$\Leftrightarrow$ normal vectors are parallel.

$$n_1 \times n_2 = 0 \Leftrightarrow \begin{vmatrix} i & j & k \\ 2 & 2 & -2 \\ 2a & 2b & -2c \end{vmatrix} = 0$$

$\Leftrightarrow \langle -4c+4b, -4a+4c, 4b-4a\rangle = 0$

$\Leftrightarrow \langle b-c, c-a, b-a\rangle = 0.$

$$\Leftrightarrow \begin{cases} b = c \\ a = c \\ a = b \\ a^2+b^2-c^2 = 1 \end{cases}$$

$\Rightarrow a^2 = 1 \Rightarrow a = \pm 1.$

$\rightsquigarrow (1,1,1), (-1,-1,-1).$

OH : none today; moved to th

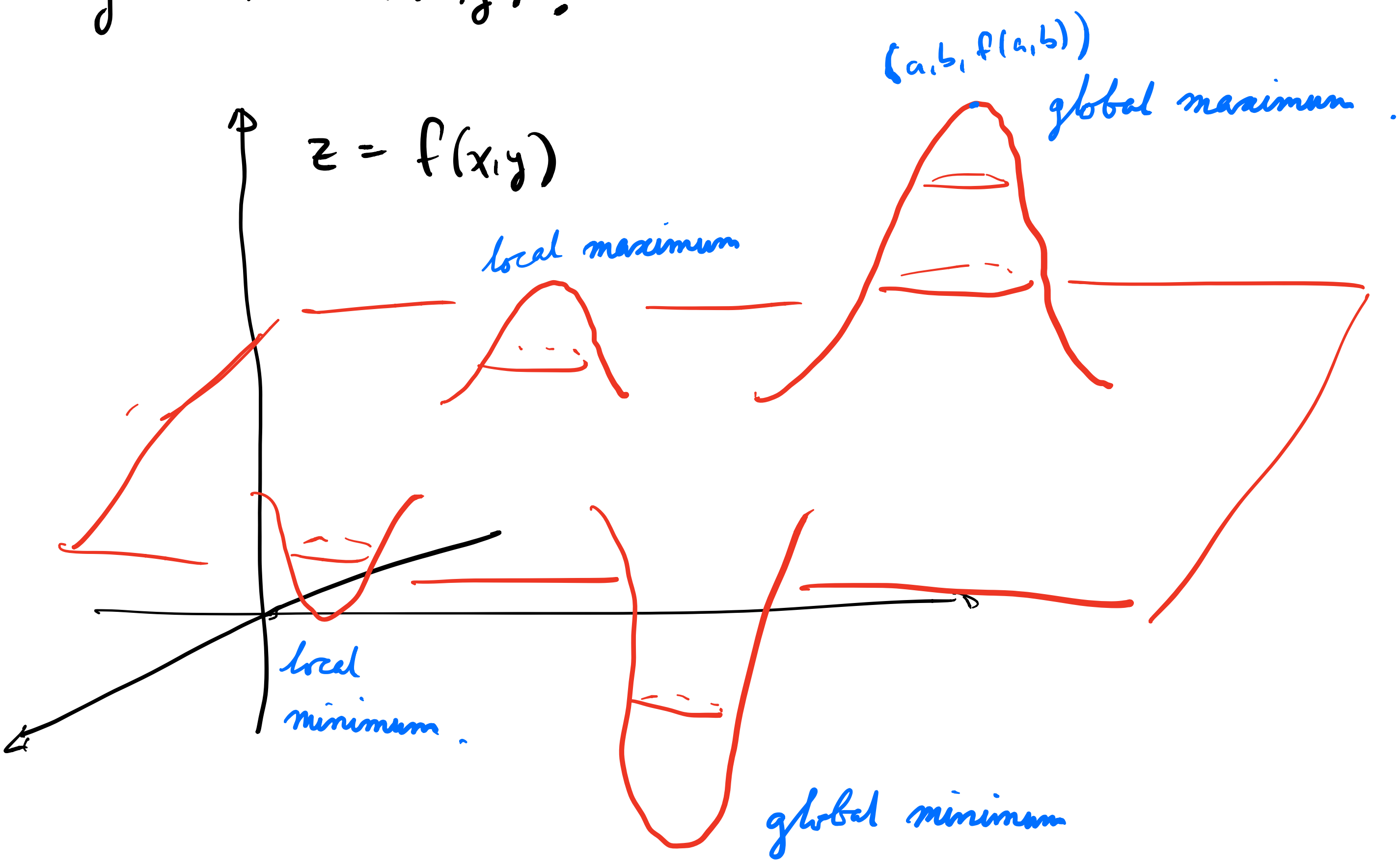
<u>Midterm 1</u> : to get curved score, $f(\text{raw}) = 0.75 \cdot \text{raw} + 10.875$.

median $= 33.75$, STD $= 5.2$

($\sim 84.4\%$)

§11.7 : Maximum, minimum.

<u>Today</u> : studying local, global max / min values of $f = f(x,y)$.



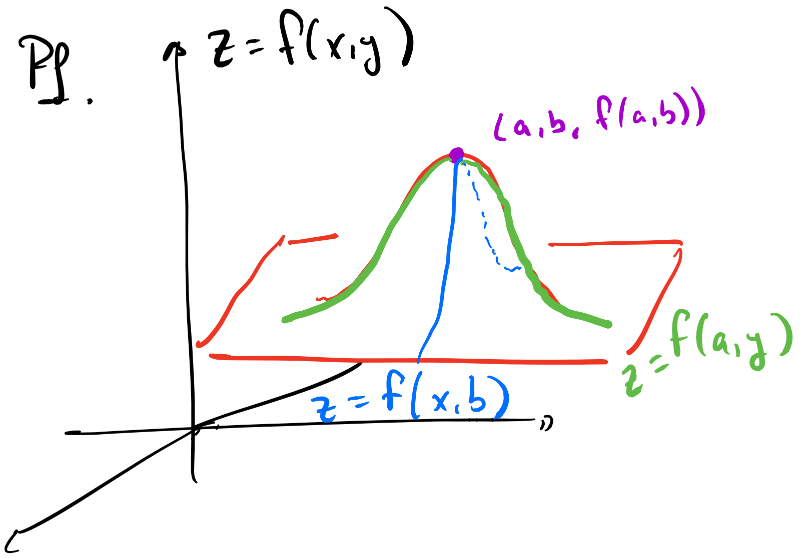
Def. $f(x,y)$ has a <u>local max</u> @ (a,b) if for all (x,y) <u>close</u> to (a,b), $f(a,b) \geq f(x,y)$. (*) If (*) holds for <u>all</u> (x,y), then (a,b) is a <u>global max</u>.

Analogously for local, global min.

max / min = "extremum".

Thm: If $f(x,y)$ has a local extremum @ (a,b), then $f_x(a,b) = 0$, $f_y(a,b) = 0$.

Pf.



$f(a,y)$ has a local max @ $y = b$
$\Rightarrow f_y(a,b) = 0$.

Similarly, $f_x(a,b) = 0$.

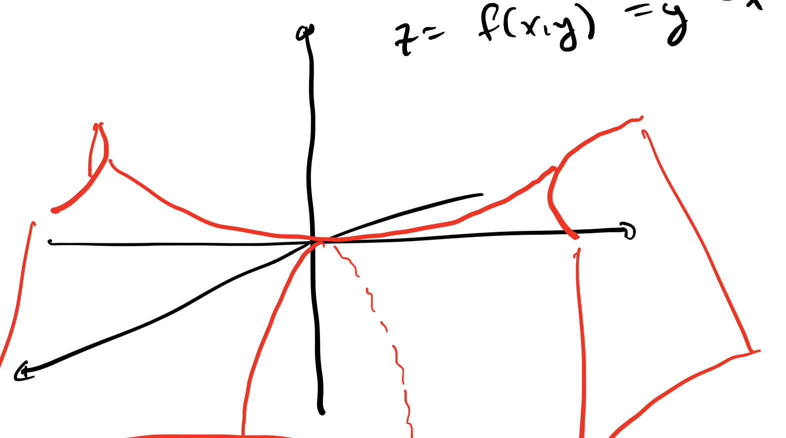
Thm $\Rightarrow$ tangent plane is horizontal @ local extremum.

Def. A critical point of $f(x,y)$ is a point (a,b) with $f_x(a,b) = 0$, $f_y(a,b) = 0$.

Not every critical pt is a local extremum!

$z = f(x,y) = y^2 - x^2$

$f_x = -2x$
$f_y = 2y$ $\Rightarrow$ $(0,0)$ is a critical pt!



say also (a,b) not in the bdry of the domain.

Second derivatives test. Say (a,b) is a critical pt of $f(x,y)$.

$$D := \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix} = \left[f_{xx}f_{yy} - f_{xy}^2\right](a,b).$$

- if $D > 0$, $f_{xx}(a,b) > 0$, local min (eg. $f = x^2+y^2$)
- if $D > 0$, $f_{xx}(a,b) < 0$, local max. (eg. $f = -x^2-y^2$)
- if $D < 0$, saddle. (eg. $f = y^2 - x^2$)

By definition, saddle = crit pt that's not a local extremum.

Ex 3. Local extrema, saddle pts of $f = x^4 + y^4 - 4xy + 1$?

a. Critical pts?

$$f_x = 4x^3 - 4y$$
$$f_y = 4y^3 - 4x.$$

Crit pts: $\begin{cases} x^3 - y = 0 & (1) \\ y^3 - x = 0. & (2) \end{cases}$

$(1) \Rightarrow y = x^3$

$(2) \Rightarrow (x^3)^3 - x = 0$

$\Leftrightarrow x^9 - x = 0$

$\Leftrightarrow x(x^8 - 1) = 0$

$\Leftrightarrow x(x^4+1)(x^4-1) = 0$

$\Leftrightarrow x(x^4+1)(x^2+1)(x^2-1) = 0$

$\Leftrightarrow x(x^4+1)(x^2+1)(x+1)(x-1) = 0.$

$\rightsquigarrow x = -1, 0, 1.$

$y = x^3 \Rightarrow$ crit pts are $(-1,-1), (0,0), (1,1)$.

$f = x^4 + y^4 - 4xy + 1.$

$f_x = 4x^3 - 4y \Rightarrow f_{xx} = 12x^2$

$f_{xy} = -4$

$f_y = 4y^3 - 4x \Rightarrow f_{yy} = 12y^2.$

$$D(-1,-1) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144 - 16 = 128$$

$f_{xx}(-1,-1) = 12 > 0$

local min!

$$D(0,0) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16.$$

saddle.

$$D(1,1) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 128.$$

$f_{xx}(-1,-1) = 12 > 0$

local min.

$\triangle$

Ex 4. Shortest distance from $(1,0,-2)$ to $x+2y+z=4$.

a. $x+2y+z=4 \Rightarrow z = -x-2y+4.$

distance of $(x, y, -x-2y+4)$ to $(1,0,2)$:

$$d = \sqrt{(x-1)^2 + y^2 + (-x-2y+4-2)^2}$$

Let's minimize the inside of the square root:

$$(x-1)^2 + y^2 + (-x-2y+2)^2 =: f(x,y).$$

Critical pts?

$$f_x = 2(x-1) + 2(-x-2y+2)(-1)$$
$$= 2x-2 + 2x + 4y - 4$$
$$= 4x + 4y - 6.$$
$$f_y = 2y + 2(-x-2y+2)(-2)$$
$$= 2y + 4x + 8y - 8$$
$$= 4x + 10y - 8.$$

$$\begin{cases} 4x+4y-6=0 \\ 4x+10y-8=0 \end{cases} \iff \begin{cases} 2x+2y=3 & (1) \\ 2x+5y=4. & (2) \end{cases}$$

$$(1) \Rightarrow y = -x + \frac{3}{2}.$$

$$(2) \Rightarrow 2x + 5\left(-x+\frac{3}{2}\right) = 4 \Rightarrow -3x + \frac{15}{2} = \frac{8}{2}$$
$$\Rightarrow -3x = -\frac{7}{2}$$
$$\Rightarrow x = 7/6.$$

$$y = -x + \frac{3}{2} = -\frac{7}{6} + \frac{9}{6}$$
$$= \frac{2}{6} = \frac{1}{3}.$$

OH: today, 1-2:30; Th, 4-5:30, F, 2-3

114 226

§11.7: Maximum, minimum values (cont'd).

Last time:

- Say $f(x,y)$ is defined near (a,b), and f has a local min / max @ (a,b). Then (a,b) is a critical pt of f.

- Second derivatives test: Say $f(x,y)$ defined near (a,b), and (a,b) a crit. pt. of f,

$$D := \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}.$$

Then:
- if $D > 0$, $f_{xx}(a,b) > 0$: f has a local min @ (a,b)
- if $D > 0$, $f_{xx}(a,b) < 0$: local max
- if $D < 0$: neither

("saddle point")

Last time, left off in middle of ...

Ex 4: Shortest distance from $(1,0,-2)$ to $x+2y+z=4$?

A. $x+2y+z=4 \Longrightarrow z=-x-2y+4$.

Distance from $(x,y,-x-2y+4)$ to $(1,0,-2)$:

$$d=\sqrt{(x-1)^2+y^2+(-x-2y+6)^2}.$$

We'll minimize $f(x,y) := (x-1)^2+y^2+(-x-2y+6)^2$.

Critical pts?

$$f_x = 2(x-1)+(-1)\cdot 2(-x-2y+6)$$
$$= 4x+4y-14 \longrightarrow f_{xx}=4, \quad f_{xy}=4$$

$$f_y = 2y+(-2)\cdot 2\cdot(-x-2y+6)$$
$$= 4x+10y-24 \longrightarrow f_{yy}=10$$

$$\begin{cases} 4x+4y-14=0 \\ 4x+10y-24=0 \end{cases} \rightsquigarrow \text{crit pt: } \left(\tfrac{11}{6}, \tfrac{5}{3}\right).$$

$$D = \begin{vmatrix} 4 & 4 \\ 4 & 10 \end{vmatrix} = 40-16 = 24 > 0$$

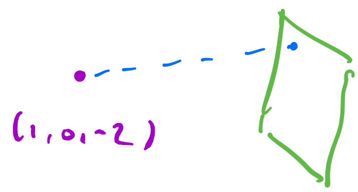
$$f_{xx} = 4 > 0 \quad \Longrightarrow \text{local min @ } \left(\tfrac{11}{6}, \tfrac{5}{3}\right).$$

shortest distance = $d=\sqrt{\left(\frac{11}{6}-1\right)^2+\left(\frac{5}{3}\right)^2+\left(-\frac{11}{6}-2\cdot\frac{5}{3}+6\right)^2}$

$$=\frac{5\sqrt{6}}{6}.$$

$\triangle$

(Idea: "geometrically clear" that there is a unique closest point on P to $(1,0,-2)$.
$(a,b,-a-2b+4)$



f has a local min @ (a,b) $\Rightarrow$ f has crit pt @ (a,b).
$\left(\frac{11}{6},\frac{5}{3}\right)$ is the only crit pt of f $\Rightarrow$ $(a,b)=\left(\frac{11}{6},\frac{5}{3}\right)$.)

Global mins, maxes.

Extreme value theorem. Say $f(x,y)$ is continuous, domain is closed, bounded. Then f has a global max, min somewhere in its domain.

includes its boundary.


✓

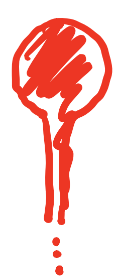
$\{x^2+y^2\leq 1\}$


✗

$\{x^2+y^2<1\}$

"does not go to infinity"


✓


✗

Strategy to find global max / min of continuous function on closed / bounded domain. K

(1) Find values of f @ critical pts in K.

(2) Find extreme values on boundary of K.

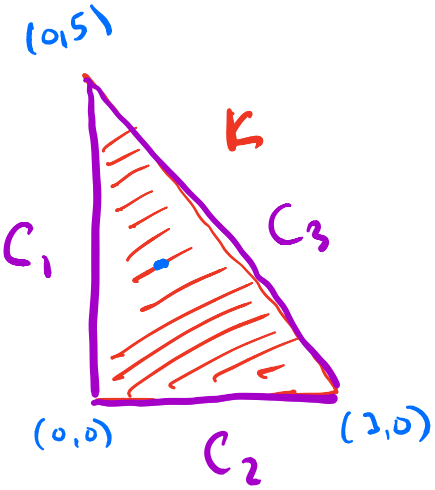
(3) Find largest, smallest values from (1), (2).

Fall '12, #7. Find absolute max, min of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region w/ vertices $(0,0)$, $(3,0)$, $(0,5)$.

a.



1. Values @ critical pts?

$f_x = 3y - 6$, $f_y = 3x - 3$

$\Longrightarrow$ crit pt is $(1,2)$.

$f(1,2) = 6 - 6 - 6 + 7$
$= 1$.

2. Extreme on bdry? Parametrize boundary pieces!

(parametrization of segment from $\underline{p}$ to $\underline{q}$ is

$$\underline{r}(t) := (1-t)\underline{p} + t\underline{q}, \quad 0 \le t \le 1.)$$

$$f(x,y) = 3xy - 6x - 3y + 7$$

- C_1: $\quad \underline{r}_1(t) := \langle 0, t \rangle, \quad 0 \le t \le 5.$

$$f(\underline{r}_1(t)) = f(0,t) = -3t + 7, \quad 0 \le t \le 5.$$

max is @ $t=0$: $\quad f(0,0) = 7$

min is @ $t=5$: $\quad f(0,5) = -8$

- C_2: $\quad \underline{r}_2(t) := \langle t, 0 \rangle, \quad 0 \le t \le 3.$

$$f(\underline{r}_2(t)) = -6t + 7.$$

max @ $t = 0$: $\quad 7. \qquad$ (@ $(0,0)$)

min @ $t = 3$: $\quad f(3,0) = -11.$

- C_3: $\quad \underline{r}_3(t) := (1-t) \cdot \langle 0,5 \rangle + t \langle 3,0 \rangle$

$$= \langle 3t, -5t+5 \rangle, \quad 0 \le t \le 1.$$

$$f(\underline{r}_3(t)) = f(3t, -5t+5) = 3 \cdot 3t \cdot (-5t+5) - 6 \cdot 3t - 3 \cdot (-5t+5) + 7$$

$$= -45t^2 + 42t - 8, \quad 0 \le t \le 1.$$

$$\frac{d}{dt}\left(-45t^2 + 42t\right) = -90t + 42$$

$$\Rightarrow 0 \text{ @ } t = \frac{42}{90} = \frac{21}{45} = \frac{7}{15}.$$