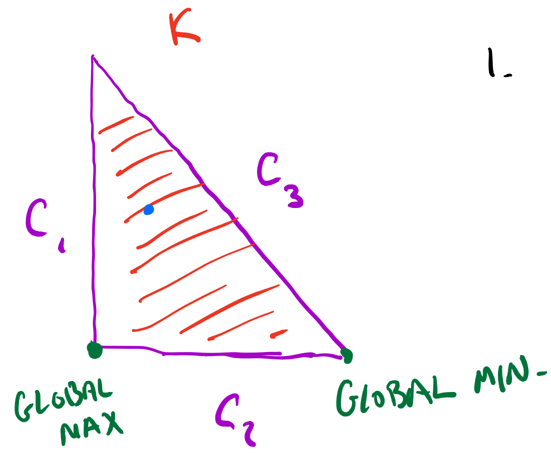
OH today: 2-3.

Fall '12, # 7. Find absolute max, min of $f = 3xy - 6x - 3y + 7$ on triangle w/ vertices $(0,0)$, $(3,0)$, $(0,5)$.

a.



1. Values @ critical pts in K?

$f(1,2) = 1.$

2. Extrema on boundary?

- C_1: $\underline{r}_1(t) = \langle 0, t \rangle$, $0 \le t \le 5$.

$g_1(t) := f(\underline{r}_1(t)) = f(0,t) = -3t + 7.$

$g_1(0) = 7$

$g_1(5) = -8!$

- C_2: $\underline{r}_2(t) = \langle t, 0 \rangle$, $0 \le t \le 3$.

$g_2(t) = f(\underline{r}_2(t)) = -6t + 7$, $0 \le t \le 3$.

$g_2(0) = 7$, $g_2(3) = -11$.

(recipe to parametrize line segment from $\underline{p}$ to $\underline{q}$: $\underline{r}(t) := (1-t)\underline{p} + t \cdot \underline{q}$, $0 \le t \le 1$.)

• C_3: $\underline{r}_3(t) = (1-t)\langle 0,5\rangle + t\cdot\langle 3,0\rangle, \quad 0 \le t \le 1.$

$$= \langle 3t, -5t+5\rangle.$$

$g(t) = f(\underline{r}_3(t)) = f(3t, -5t+5) = -45t^2 + 42t - 8, \quad 0 \le t \le 1.$

$g'(t) = -90t + 42 \Rightarrow g' = 0$ @ $t = \frac{42}{90} = \frac{7}{15}.$

$g(7/15) = 9/5.$ max of g

$g(0) = -8,$ $\quad g(1) = -45 + 42 - 8 = -11.$ min of g.

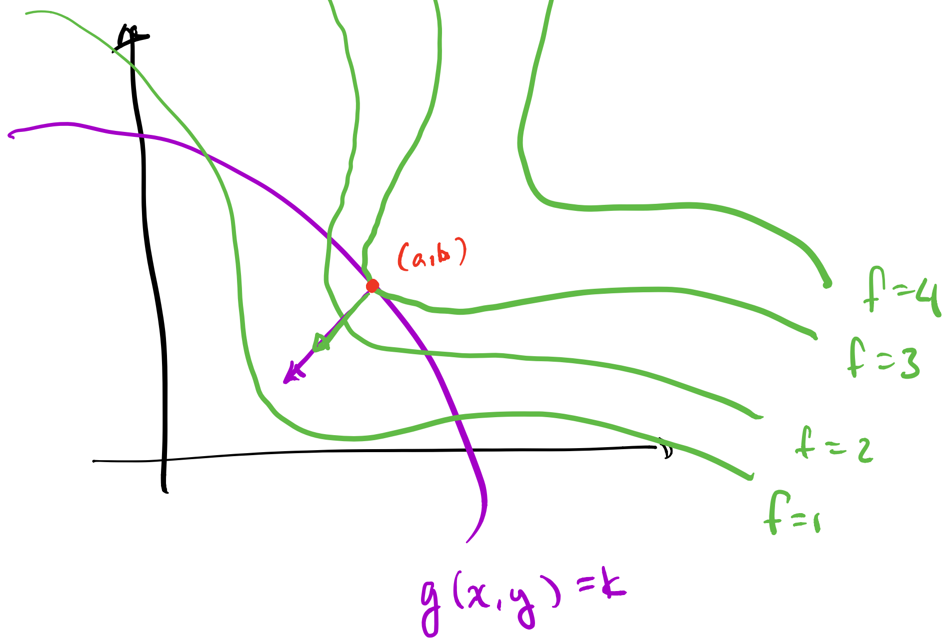
3. Our glorious list of candidates:

	GLOBAL MAX	GLOBAL MIN	
1	7, -8	7, -11	9/5, -11
f @ (1,2)	max, min on C_1	C_2	C_3

§11.8: Lagrange multipliers.

Goal here: maximize/minimize $f(x,y)$ or $f(x,y,z)$ subject to the constraint $g(x,y) = k$ or $g(x,y,z) = k$.

Geometric idea:



Method of Lagrange multipliers: to find max / min of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$:

(1) Find solutions of:
$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases}$$

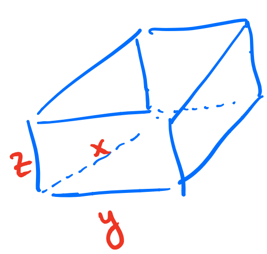
(x,y,z,λ)

(2) Evaluate f at those candidates.

* technical note: only valid when these extrema exist, and when $\nabla g \neq \underline{0}$ on $\{g=k\}$.

Spring '15, #4. Want to make rectangular box w/o lid, w/ 48 in^2 of material. Find max volume.

a.



$$\underbrace{2xz + 2yz + xy}_{g} = 48.$$

$$f(x,y,z) = xyz.$$

$$\nabla f = \langle yz,\ xz,\ xy \rangle; \qquad \nabla g = \langle 2z+y,\ 2z+x,\ 2x+2y \rangle.$$

$$\rightsquigarrow \begin{cases} yz = \lambda(2z+y) & (1) \\ xz = \lambda(2z+x) & (2) \\ xy = \lambda(2x+2y) & (3) \\ 2xz+2yz+xy = 48. & (4) \end{cases}$$

(note: $\lambda \neq 0$, o/w $xy = xz = yz = 0$)

$$(1) \Rightarrow xyz = \lambda x(2z+y)$$
$$(2) \Rightarrow xyz = \lambda y(2z+x)$$
$$(3) \Rightarrow xyz = \lambda z(2x+2y)$$

$$\Rightarrow \lambda x(2z+y) = \lambda y(2z+x) = \lambda z(2x+2y)$$

$$\Rightarrow \underbrace{2xz+xy}_{\alpha} = \underbrace{2yz+xy}_{\beta} = \underbrace{2xz+2yz}_{\gamma}.$$

$$\underline{\alpha=\beta} \xRightarrow{-xy} 2xz = 2yz = z(x-y) = 0$$

$$\Rightarrow z=0 \text{ or } \boxed{x=y}.$$

impossible: $z=0 \Rightarrow xyz=0$.

$$\underline{\alpha=\gamma} \xRightarrow{-2xz} xy = 2yz \Rightarrow y(x-2z) = 0$$

$$\Rightarrow \cancel{y=0} \text{ or } \boxed{z = \frac{x}{2}}$$

$$\Rightarrow y = x, \quad z = x/2.$$

(4): $2xz + 2yz + xy = 48$

$$\Rightarrow x^2 + x^2 + x^2 = 48 \Rightarrow 3x^2 = 48$$
$$\Rightarrow x^2 = 16$$
$$\Rightarrow x = 4$$

$\Rightarrow$ $(x, y, z) = (4, 4, 2)$,

volume of 32.

OH this week: M 12-1:30, W 1-2:30, F 2-3

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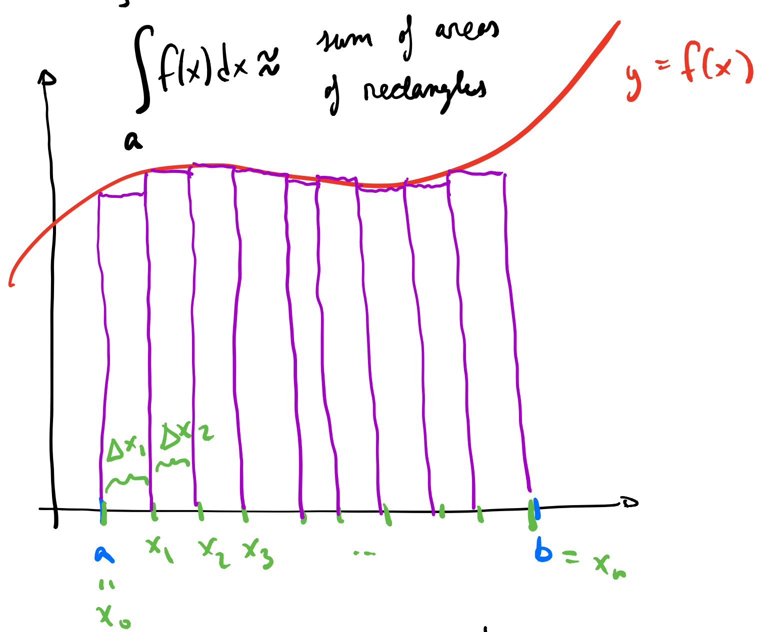
§12.1: Double integrals over rectangles.

Recall the definition of the definite integral $\int_a^b f(x)\,dx$.

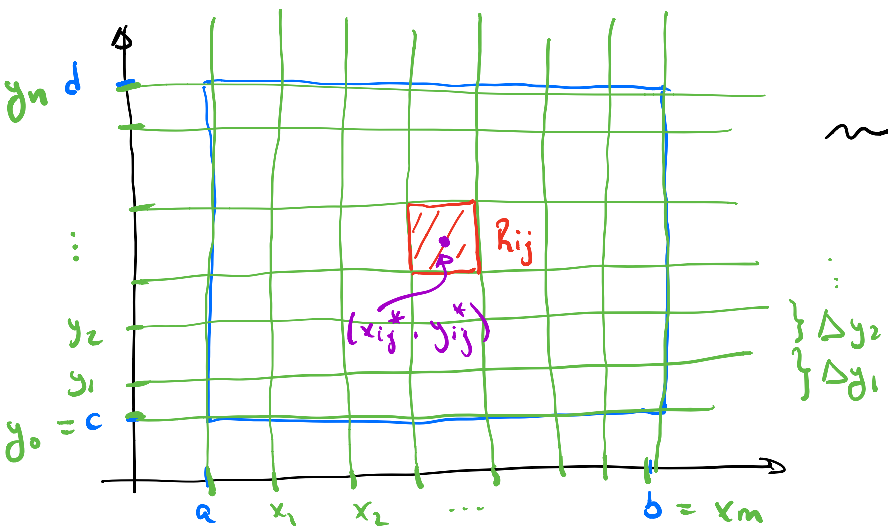


$$\int_a^b f(x)\,dx := \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

"sample point" $x_{i-1} \le x_i^* \le x_i$.

When $f(x) \geq 0$, $\int_a^b f(x)\,dx$ is the area under the graph $y = f(x)$, $a \le x \le b$.

Double integrals over rectangles. Using "volume under graph" as a guide, let's define $\iint_R f(x,y)\,dA$, $R = [a,b] \times [c,d] = \left\{ \begin{array}{l} a \le x \le b, \\ c \le y \le d \end{array} \right\}$.



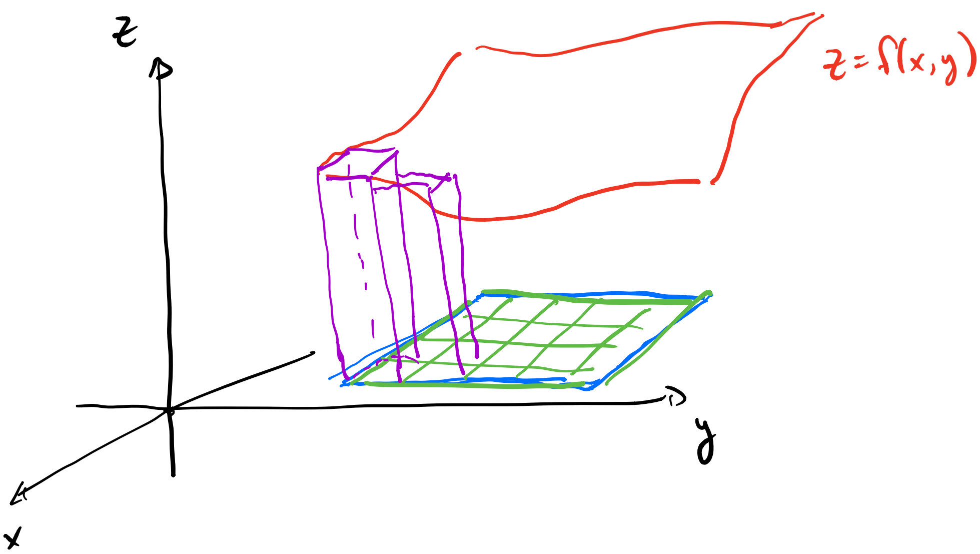
$\rightsquigarrow$ $\boxed{\text{Def. } \iint_R f(x,y)\,dA := \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1, j=1}^{i=m, j=n} f(x_{ij}^*, y_{ij}^*)\,\Delta x_i\, \Delta y_j}$

(exists when f is continuous)

$\overset{\|}{x_0}$

$\Delta x_1, \Delta x_2 \ldots$

Fact: If $f(x,y) \geq 0$, then $\iint_R f(x,y)\,dA$ is the volume below the graph $z = f(x,y)$.



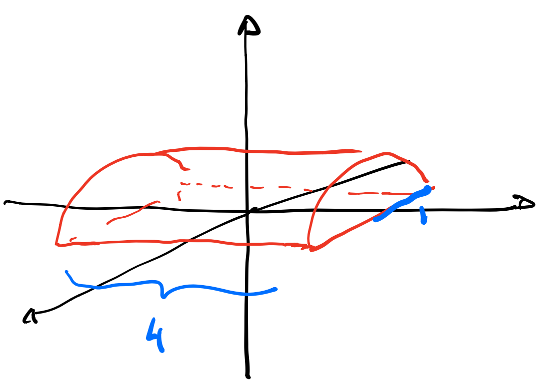
Ex 2. $R = [-1,1] \times [-2,2]$. Compute $I = \iint_R \sqrt{1-x^2}\,dA$.

a. Note, $\sqrt{1-x^2} \geq 0$ on $R \Rightarrow I$ is the volume under the graph $z = \sqrt{1-x^2}$.

Ie, $I = $ volume of $\left\{ \begin{array}{l} -1 \leq x \leq 1 \\ -2 \leq y \leq 2 \\ 0 \leq z \leq \sqrt{1-x^2} \end{array} \right\}$.

$z = \sqrt{1-x^2} \Rightarrow z^2 = 1 - x^2 \Rightarrow x^2 + z^2 = 1$.

circular cylinder of radius 1, centered on y-axis.



$I = \text{length} \cdot \text{area of cross-section} = 4 \cdot \left(\frac{1}{2} \cdot \pi \cdot 1^2\right)$

$= 2\pi.$

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How to compute double integrals more generally?

$\underline{\text{Fubini's theorem}}$. Say $f(x,y)$ continuous on $R=[a,b]\times[c,d]$.

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

"iterated integrals"

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

$\underline{\text{Ex}}$.

$$\int_0^1\int_1^3 xe^{xy}\,dy\,dx = \int_0^1\left(\int_{u=x}^{u=3x} \underbrace{xe^u\cdot\frac{1}{x}}_{=e^u}\,du\right)dx$$

$u=xy$

$du = x\,dy$

$\Rightarrow dy = \frac{1}{x}du$

$$= \int_0^1 \left[e^u\right]_{u=x}^{u=3x}\,dx$$

$$= \int_0^1 \left(e^{3x}-e^x\right)dx$$

$$= \left[\frac{1}{3}e^{3x} - e^x\right]_{x=0}^{x=1}$$

$$= \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3}-1\right)$$

$$\boxed{= \frac{1}{3}e^3 - e + \frac{2}{3}.}$$

$\triangle$

Ex 7 Volume of solid S bounded by:

- coordinate planes
- $\{x=2\}$, $\{y=2\}$
- $\{x^2+2y^2+z=16\}$ $\Leftrightarrow z = 16 - x^2 - 2y^2$?

$$S = \left\{ \begin{array}{ccccc} 0 & \le & x & \le & 2 \\ 0 & \le & y & \le & 2 \\ 0 & \le & z & \le & 16-x^2-2y^2 \end{array} \right\}.$$

≥ 0 on $\left\{ \begin{array}{c} 0 \le x \le 2 \\ 0 \le y \le 2 \end{array} \right\}$.

$$\Rightarrow \text{volume of } S = \int_0^2\int_0^2 (16-x^2-2y^2)\,dy\,dx.$$

$$= \int_0^2 \left[16y - x^2y - \frac{2}{3}y^3\right]_{y=0}^{y=2} dx$$

$$= \int_0^2 \Big(\underbrace{32}_{=\frac{96}{3}} - 2x^2 - \underbrace{\frac{2}{3}\cdot 8}_{=\frac{16}{3}}\Big)\,dx$$

$$= \int_0^2 \left(\frac{80}{3} - 2x^2\right) dx$$

$$= \left[\frac{80}{3}x - \frac{2}{3}x^3\right]_{x=0}^{x=2}$$

$$= \frac{160}{3} - \frac{16}{3} = \frac{144}{3} = 48.$$

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OH this week: W 1-2:30, F 2-3

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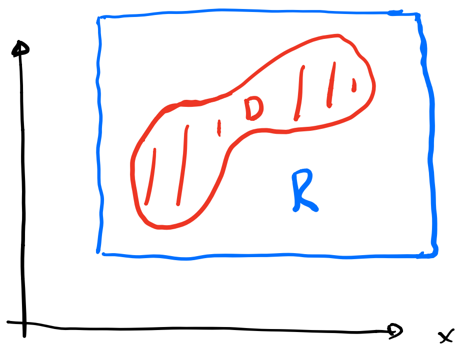
§12.2: Double integrals over more general regions.

What if we want to integrate $f(x,y)$ over a region D that's not a rectangle?

bounded

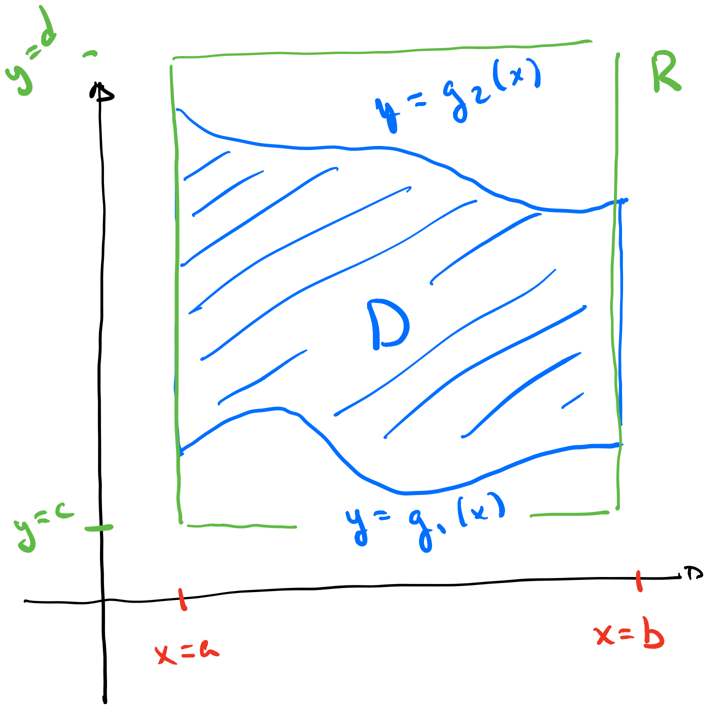


- choose R that contains D
- Define $F(x,y)$ on R, by: $F(x,y) := \begin{cases} f(x,y), & (x,y) \in D, \\ 0, & (x,y) \in R \setminus D \end{cases}$

Now set $\boxed{\iint_D f(x,y)\, dA := \iint_R F(x,y)\, dA.}$

OK, but how do we actually compute $\iint\limits_D$??

First step: assume $D = \left\{ \begin{array}{l} a \le x \le b, \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$.



$$I = \iint\limits_D f(x,y)\,dA$$

$$= \iint\limits_R F(x,y)\,dA$$

$$= \int_a^b \left(\int_c^d F(x,y)\,dy \right) dx$$

$$= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

$\Rightarrow$ If D is the "type-1 region" $D = \left\{ \begin{array}{l} a \le x \le b, \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$,

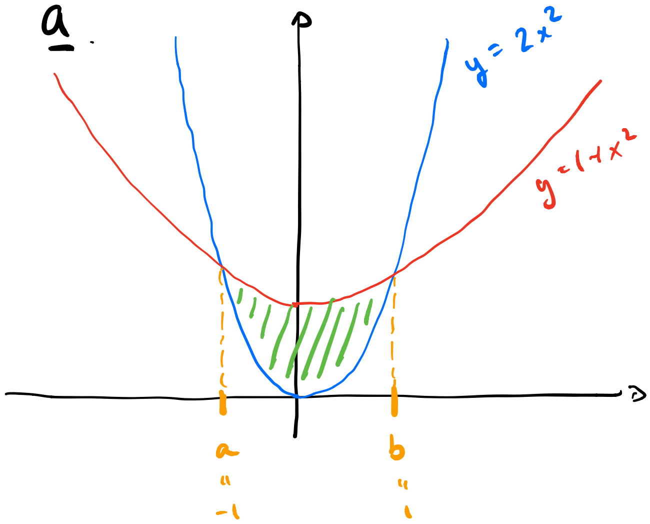
then $$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

If D is the "type-2 region" $D = \left\{ \begin{array}{l} a \le y \le b, \\ g_1(y) \le x \le g_2(y) \end{array} \right\}$,

then $$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy.$$

Ex 1. $I = \iint_D (x+2y)\, dA$, D = region bounded by $y = 2x^2$, $y = 1+x^2$.

a.



intersection pts of $y = 2x^2$, $y = 1+x^2$?

$2x^2 = 1 + x^2 \iff x^2 = 1$

$\iff x = \pm 1.$

$$\Rightarrow D = \left\{ \begin{array}{c} -1 \le x \le 1 \\ 2x^2 \le y \le 1+x^2 \end{array} \right\}.$$

type - 1

$$I = \iint_D (x+2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2 \Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \left(x(1+x^2) + (1+x^2)^2\right) - \left(x(2x^2) + (2x^2)^2\right)$$
$$= (x + x^3 + 1 + 2x^2 + x^4) - (2x^3 + 4x^4)$$
$$= x - x^3 + 1 + 2x^2 - 3x^4$$

$$= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right) dx$$

$$= \left[-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \right]_{x=-1}^{x=1}$$

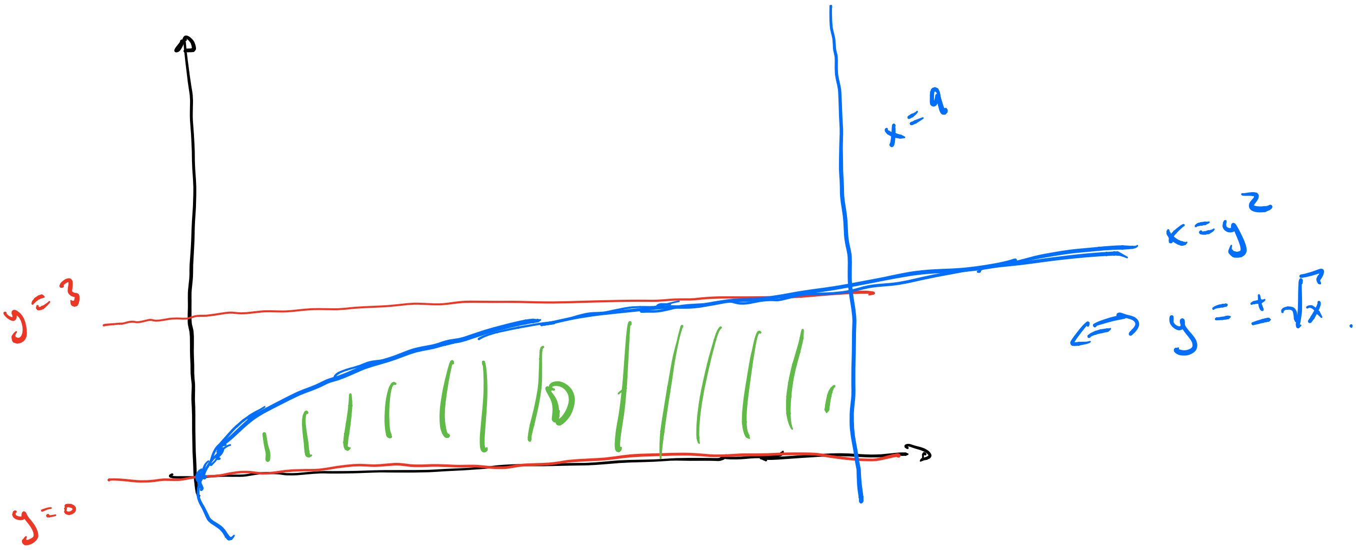
$$= 2 \cdot -\frac{3}{5} + 2 \cdot \frac{2}{3} + 2 \cdot 1$$

$$= -\frac{6}{5} + \frac{4}{3} + 2$$

$$= -\frac{18}{15} + \frac{20}{15} + \frac{30}{15} = \frac{32}{15}.$$

Fall '14, #6a. Evaluate $I = \int_0^3 \int_{y^2}^9 y \cos(x^2)\, dx\, dy$.

a. Change order of integration? $D := \left\{ \begin{array}{l} 0 \le y \le 3 \\ y^2 \le x \le 9 \end{array} \right\}$.



$$D = \left\{ \begin{array}{l} 0 \le x \le 9 \\ 0 \le y \le \sqrt{x} \end{array} \right\}$$

$$\Rightarrow \quad I = \int_0^9 \int_0^{\sqrt{x}} y \cos(x^2)\, dy\, dx$$

$$= \int_0^9 \left[\frac{1}{2} y^2 \cos(x^2) \right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^9 \frac{1}{2} x \cos(x^2)\, dx$$

$u = x^2 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2} x\, dx = \frac{1}{4} du$

$$= \int_0^{81} \frac{1}{4} \cos u\, du = \left[\frac{1}{4} \sin u \right]_{u=0}^{u=81} = \frac{1}{4} \sin(81).$$

Δ

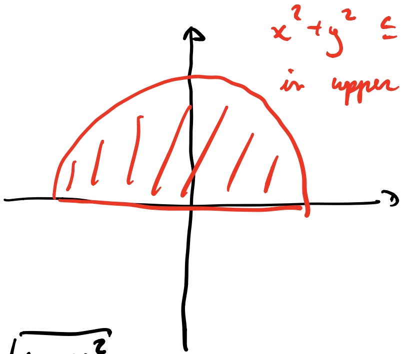
OH today: 2 - 3.

§12.3: Double in polar coordinates.

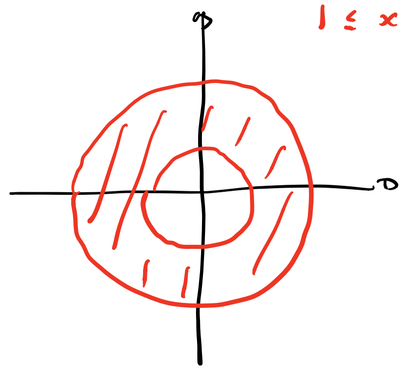
r, θ instead of x, y.

Goal: better way to integrate over regions like:

$x^2+y^2 \le 1$, in upper half-plane



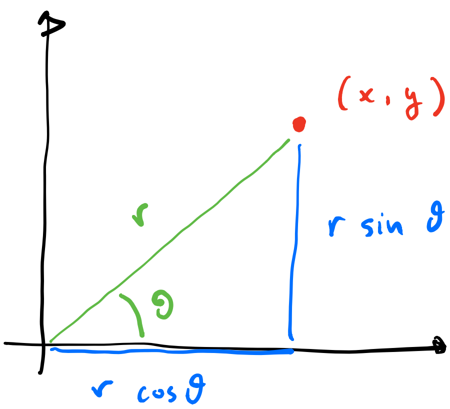
$1 \le x^2+y^2 \le 4$

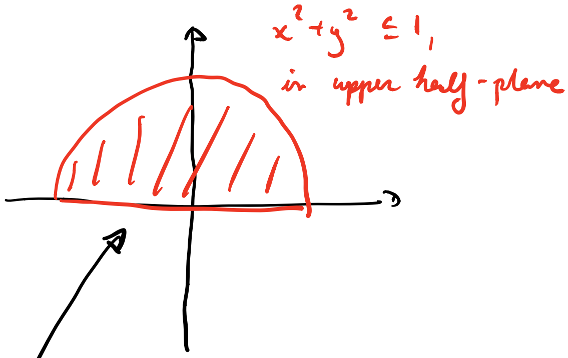


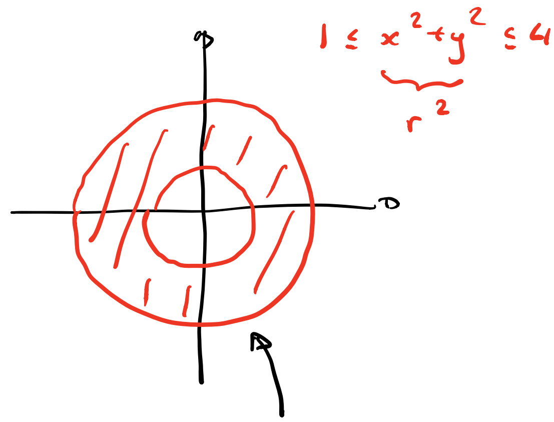
$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$$

$$\boxed{\text{Polar coordinates: } x = r\cos\theta, \quad y = r\sin\theta}$$

$$\Longrightarrow \quad x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$





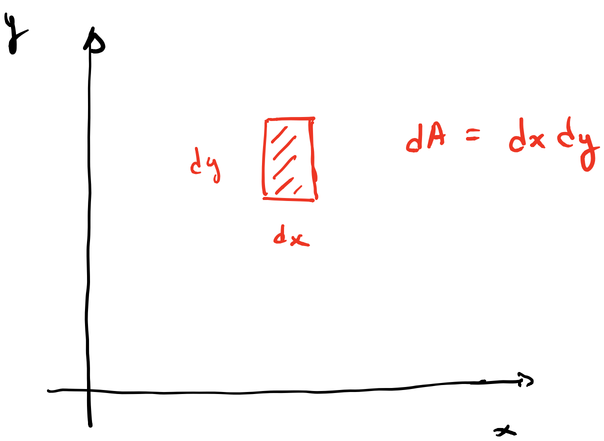


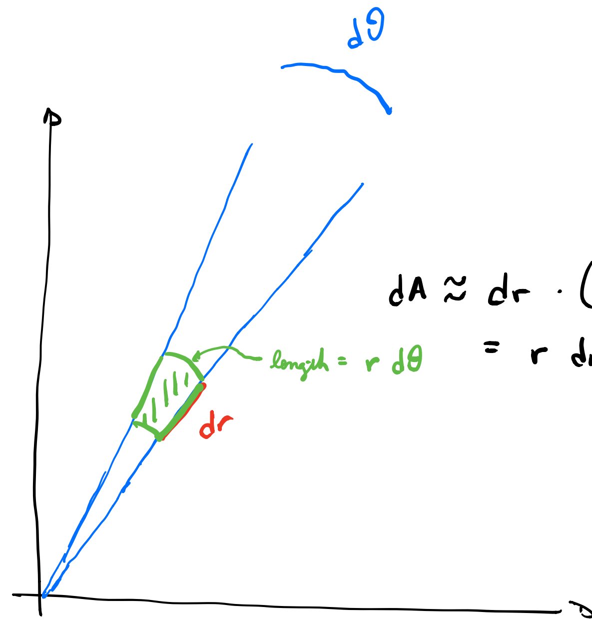
in polar coordinates: $\left\{ \begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le \pi \end{array} \right\}$ $\qquad$ $\left\{ \begin{array}{l} 1 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array} \right\}$

"polar rectangle"

Integrating in polar coordinates.

How to rewrite dA?





$$dA \approx dr \cdot (r\, d\theta) = r\, dr\, d\theta.$$

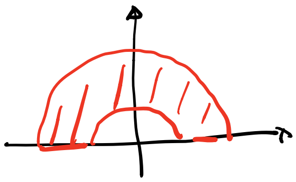
So: Say $R = \left\{ \begin{array}{l} a \le r \le b \\ \alpha \le \theta \le \beta \end{array} \right\}$. then:

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b \left[f(r\cos\theta, r\sin\theta)\, r \right] dr\, d\theta.$$

$$= \int_a^b \int_\alpha^\beta r\, f(r\cos\theta, r\sin\theta)\, d\theta\, dr.$$

(can also integrate in the other order)

Ex 1. $I = \iint_R (3x+4y^2)\, dA$, R = region bounded by $x^2+y^2=1$, $x^2+y^2=4$ in upper half-plane.

a. 

$$R = \left\{ \begin{array}{l} 1 \le r \le 2 \\ 0 \le \theta \le \pi \end{array} \right\},$$

$$I = \int_0^{\pi}\int_1^2 r\left(3r\cos\theta + 4r^2\sin^2\theta\right) dr\, d\theta$$

$$= \int_0^{\pi}\int_1^2 \left(3r^2\cos\theta + 4r^3\sin^2\theta\right) dr\, d\theta$$

$$= \int_0^{\pi} \left[r^3\cos\theta + r^4\sin^2\theta \right]_{r=1}^{r=2} d\theta$$

use: $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.

$$= \int_0^{\pi} \left(7\cos\theta + 15\sin^2\theta\right) d\theta$$

$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2}\left(1-\cos 2\theta\right)\right) d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\right]_0^{\pi}$$

$$= \frac{15\pi}{2}.$$

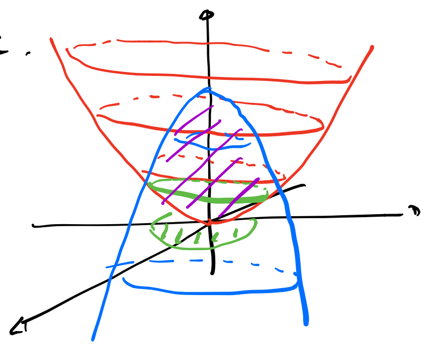
$\triangle$

Say $D = \left\{ \begin{array}{l} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \end{array} \right\}$. then:

$$\iint_D f(x,y)\, dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} r\, f(r\cos\theta, r\sin\theta)\, dr\, d\theta.$$

Fall '14, #6b. Find intersection of $\{z = x^2+y^2\}$,
$\{z = 8-3x^2-3y^2\}$.
Find volume of region between these surfaces.

a.



Intersection? $x^2+y^2 = 8-3x^2-3y^2$

$\Leftrightarrow 4x^2+4y^2 = 8$

$\Leftrightarrow x^2+y^2 = 2.$

$$\Rightarrow \boxed{\left\{\begin{array}{l} x^2+y^2 = 2, \\ z = 2 \end{array}\right\}}$$

$$V = \iint_D \overbrace{\left((8-3x^2-3y^2)-(x^2+y^2)\right)}^{=8-4(x^2+y^2)} dA$$

$$D = \{x^2+y^2 \le 2\} = \left\{\begin{array}{l} 0 \le r \le \sqrt{2} \\ 0 \le \theta \le 2\pi \end{array}\right\}$$

$$= \int_0^{\sqrt{2}} \int_0^{2\pi} r \cdot (8-4r^2)\, d\theta\, dr$$

$$= \int_0^{\sqrt{2}} \int_0^{2\pi} (8r-4r^3)\, d\theta\, dr$$

$$= \int_0^{\sqrt{2}} 2\pi(8r-4r^3)\, dr$$

$$= \left[2\pi(4r^2 - r^4)\right]_{r=0}^{r=\sqrt{2}} = 2\pi\cdot(8-4)$$

$$= 8\pi.$$

Δ

OH this week: M 12–1:30, W 1–2:30, F 2–3

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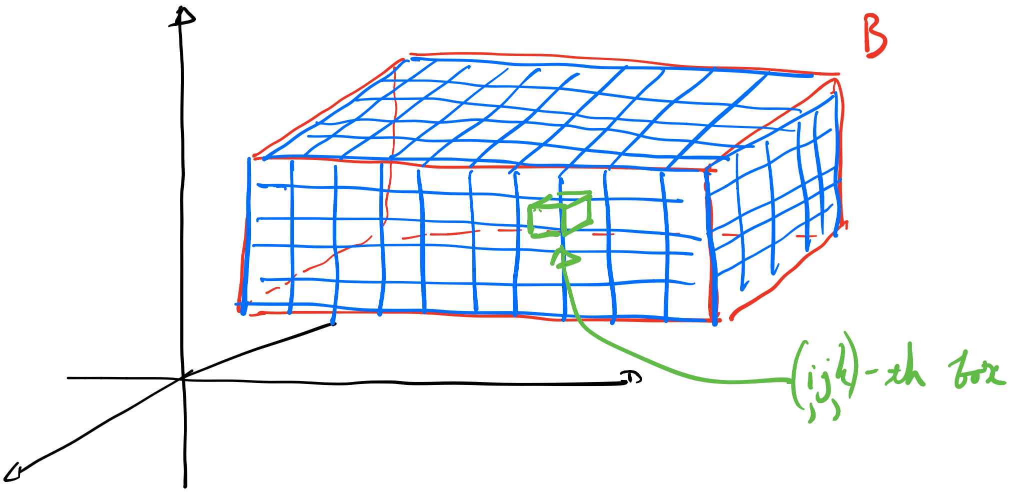
§12.5: triple integrals.

Analogous w/ $\iint$, can define $\iiint_D f(x,y,z)\, dV$

Start w/ simple case, that $D = B = \left\{ \begin{array}{l} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{array} \right\}$.



Def. $\displaystyle\iiint_B f(x,y,z)\, dV := \lim_{\max \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V_{ijk}$.

(makes sense when f is continuous)

Fubini for $\iiint$: $\displaystyle\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$

(can also use other orders of integration)

Ex 1. Compute $I = \iiint_B xyz^2 \, dV$, $B = \left\{ \begin{array}{l} 0 \le x \le 1 \\ -1 \le y \le 2 \\ 0 \le z \le 3 \end{array} \right\}$.

A.
$$I \overset{\text{Fubini}}{=} \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx \, dy \, dz$$

$$= \int_0^3 \int_{-1}^2 \left[\frac{1}{2} x^2 y z^2 \right]_{x=0}^{x=1} dy \, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} y z^2 \, dy \, dz$$

$$= \int_0^3 \left[\frac{1}{4} y^2 z^2 \right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3}{4} z^2 \, dz$$

$$= \left[\frac{1}{4} z^3 \right]_{z=0}^{z=3} = \frac{27}{4}.$$

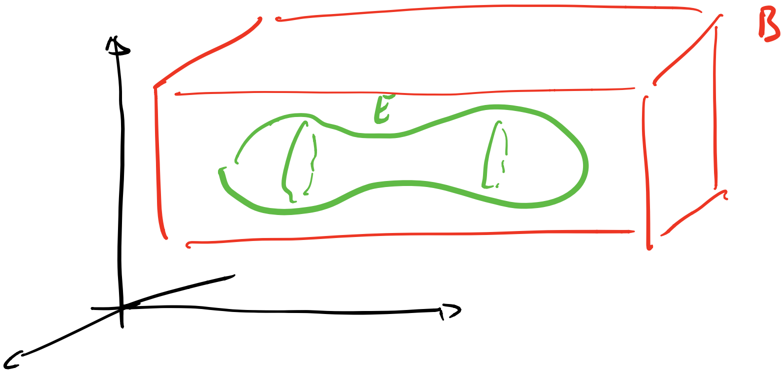
Δ

We can also integrate over general bounded regions in $\mathbb{R}^3$.

Given $f(x,y,z)$ on bounded E, choose box $B \supset E$.

Then define $F(x,y,z)$ on B, by $F(x,y,z) := \begin{cases} f(x,y,z), & (x,y,z) \in E, \\ 0, & (x,y,z) \in B \setminus E. \end{cases}$

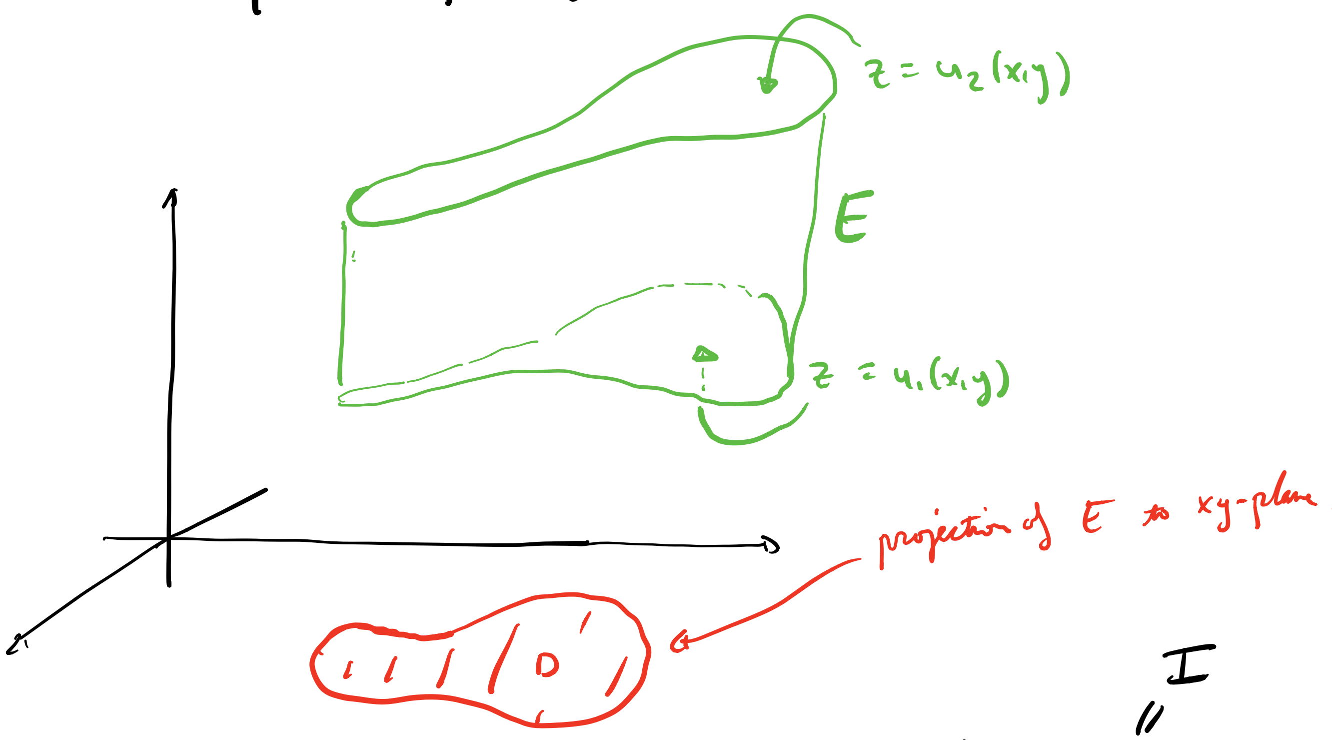
then:
$$\boxed{\iiint_E f(x,y,z) \, dV := \iiint_B F(x,y,z) \, dV.}$$



How to compute $\iiint_E$?

Start by considering a type-1 region in $\mathbb{R}^3$:

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} (x,y) \in D, \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}.$$



By similar logic to §12.2,

$$\iiint_E f(x,y,z)\, dV = \underbrace{\iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \, dA}_{=\, I}.$$

Now, if $D = \left\{ \begin{array}{l} a \le x \le b, \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$, then

$$I = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

⇝ if $E = \left\{ \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$, then:

$$\iiint_E f(x,y,z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

(similar formula if x, y, z appear in different order)

<u>Fall '15, #5</u>. E: region in first octant, below $\{6x+3y+z=6\}$.

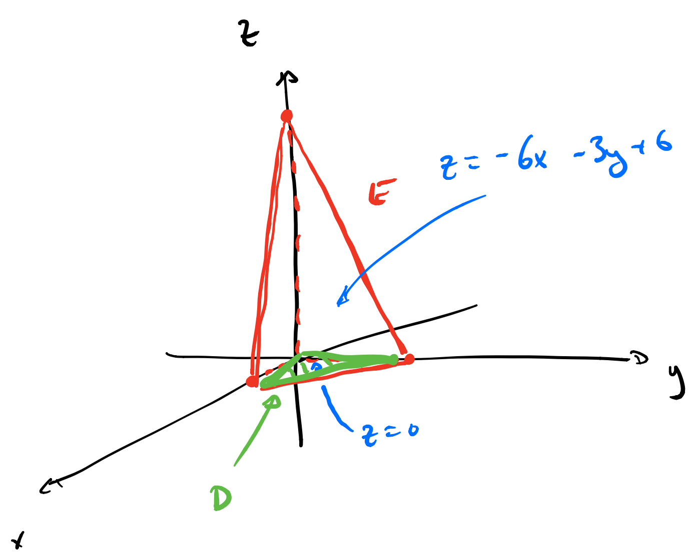
(a) Express $I = \iiint_E f(x,y,z)\, dV$ as iterated integral w/ order $dz\, dx\, dy$.

(b) Same, but $dx\, dy\, dz$.

<u>A</u>. (a) Want $I = \int_a^b \int_{g_1(y)}^{g_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dx\, dy$.

So, need to write E in the form $E = \left\{ \begin{array}{l} \overset{0}{a} \le y \le \overset{2}{b} \\ \underset{0}{g_1(y)} \le x \le g_2(y) \\ \underset{0}{u_1(x,y)} \le z \le \underset{-6x-3y+6}{u_2(x,y)} \end{array} \right\}$.

$-\frac{1}{2}y+1$



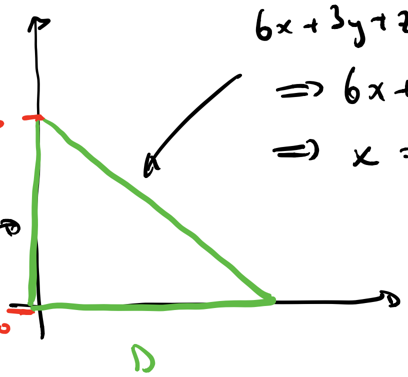
E: region in first octant, below $\{6x+3y+z=6\}$.

x-, y-, z- intercepts are $x=1$, $y=2$, $z=6$.

What's D?

$6x + 3y = 6,\ x = 0$
$\Rightarrow 3y = 6 \Rightarrow y = 2$

$x = 0$



$6x + 3y + z = 6,\ z = 0$

$\Rightarrow 6x + 3y = 6$

$\Rightarrow x = -\frac{1}{2}y + 1.$

$$E = \left\{ \begin{array}{c} \overset{0}{a} \le y \le \overset{2}{b} \\ \underset{0}{g_1(y)} \le x \le g_2(y) \quad -\tfrac{1}{2}y+1 \\ \underset{0}{u_1(x,y)} \le z \le \underset{-6x-3y+6}{u_2(x,y)} \end{array} \right\}.$$

$$\Rightarrow \boxed{I = \int_0^2 \int_0^{-\frac{1}{2}y+1} \int_0^{-6x-3y+6} f(x,y,z)\, dz\, dx\, dy.}$$

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OH this week: W 1-2:30, F 2-3

114

§12.6: Cylindrical coordinates.

Cylindrical coordinates: transform $x, y, z \rightsquigarrow r, \theta, z$.

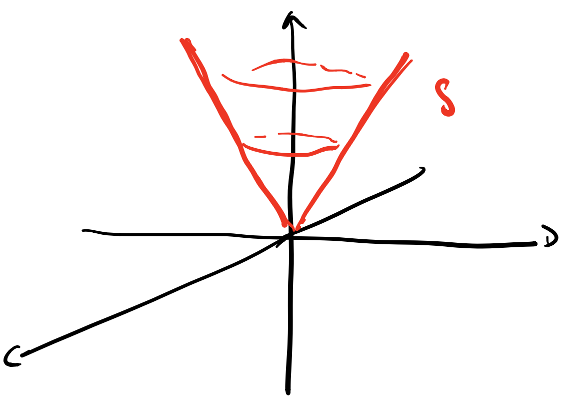
$(x = r\cos\theta,\ y = r\sin\theta,\ z = z)$

Ex 2 Consider surface $z = r$ (S). Describe it.

A. $z = r \Rightarrow z = \sqrt{x^2+y^2}$

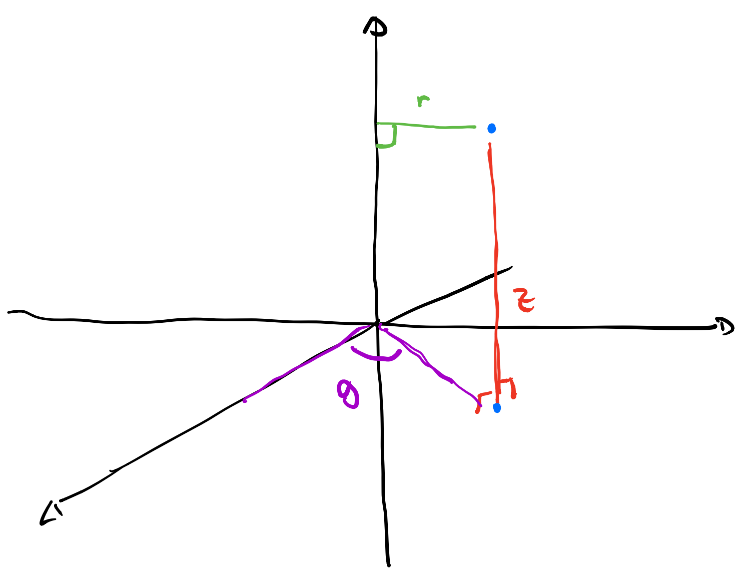
$\Rightarrow z^2 = x^2+y^2$.

z-slice? $z^2 = x^2+y^2 \overset{z=a}{\rightsquigarrow} a^2 = x^2+y^2$ (radius-$|a|$ circle centered @ $(0,0)$).



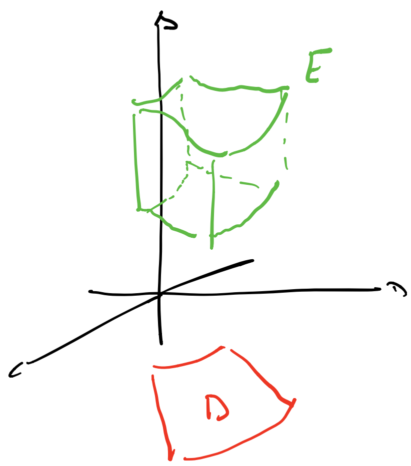
(note: r = distance to z-axis)

$\triangle$



How to rewrite $\iiint_E$ in cylindrical coordinates?

This is a good idea if E has rotational symmetry about z-axis, and/or if x^2+y^2 shows up in integrand.



$$E = \left\{ \begin{array}{l} (x,y) \in D, \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}, \quad D = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \end{array} \right\}$$

$$I = \iiint_E f(x,y,z)\, dV$$

$$= \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right) dA$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} r \left(\int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz \right) dr\, d\theta.$$

Suppose: $E = \left\{ \begin{array}{l} (x,y) \in D, \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}, \quad D = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \end{array} \right\}$

Then: $$\iiint_E f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} r\, f(r\cos\theta, r\sin\theta, z)\, dz\, dr\, d\theta.$$

Ex 4. Compute $I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$

using cylindrical coordinates.

a. Rewrite $I = \iiint_E (x^2+y^2)\, dV$, $E = \left\{ \begin{array}{l} \boxed{\begin{array}{l} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{array}}^{\,D} \\ \underbrace{\sqrt{x^2+y^2}}_{z=\sqrt{x^2+y^2}} \le z \le \underbrace{2}_{z=2} \end{array} \right\}$.

intersection? $\sqrt{x^2+y^2} = 2$

$\Rightarrow x^2+y^2 = 4$

$D = \left\{ \begin{array}{l} -2 \le x \le 2 \\ \underbrace{-\sqrt{4-x^2}}_{} \le y \le \underbrace{\sqrt{4-x^2}}_{} \end{array} \right\}$.

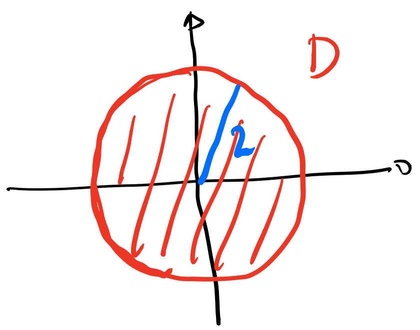
$-\sqrt{4-x^2} = y$

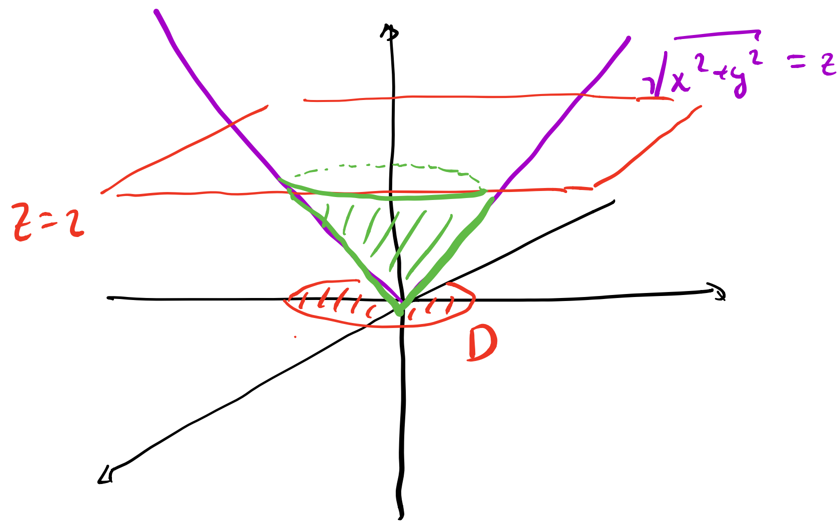
$\Rightarrow 4-x^2 = y^2$

$\Rightarrow x^2+y^2 = 4$

$y = \sqrt{4-x^2}$

$\Rightarrow x^2+y^2 = 4$





$E = \left\{ \begin{array}{l} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \\ r \le z \le 2 \end{array} \right\} \Rightarrow I = \iiint_E \overbrace{(x^2+y^2)}^{r^2}\, dV$

$= \int_0^{2\pi} \int_0^2 \int_r^2 r^3\, dz\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^2 \left[z r^3 \right]_{z=r}^{z=2} dr\, d\theta$

$$= \int_0^{2\pi}\int_0^2 (2r^3 - r^4)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5 \right]_{r=0}^{r=2} d\theta$$

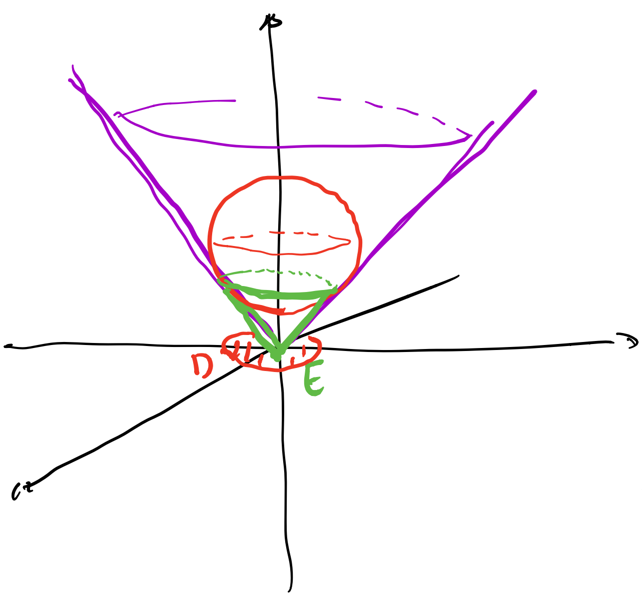
$$= \int_0^{2\pi} \underbrace{\left(8 - \tfrac{32}{5}\right)}_{=8/5} d\theta$$

$$= \frac{16}{5}\pi.$$

$\triangle$

Spring '18, 3b. E solid bounded below by $z=\sqrt{x^2+y^2}$, above by sphere $x^2+y^2+(z-2)^2=2$. Set up $\iiint$ in cylindrical coord's computes volume E.

A. (Recall: $\mathrm{Vol}(E) = \iiint_E 1\,dV$, $\mathrm{Area}(D) = \iint_D 1\,dA$.)



intersection between purple cone, red sphere?

$z=\sqrt{x^2+y^2}$ (1)

$x^2+y^2+(z-2)^2=2$ (2)

(1) → (2) $\Rightarrow z^2+(z-2)^2=2$

$\Rightarrow 2z^2-4z+2=0$

$\Rightarrow z^2-2z+1=0$

$\Rightarrow (z-1)^2=0$

$\Rightarrow z=1,\quad x^2+y^2=1$

$D = \{x^2+y^2 \le 1\}$

$= \{r \le 1\}.$

$$E = \left\{ \begin{array}{l} x^2 + y^2 \le 1 \\ \sqrt{x^2+y^2} \le z \le ? \end{array} \right\}$$

$$x^2 + y^2 + (z-2)^2 = 2$$

$$\Rightarrow (z-2)^2 = 2 - x^2 - y^2$$

$$\Rightarrow z = 2 \pm \sqrt{2 - x^2 - y^2}$$

$$= \left\{ \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ r \le z \le 2 - \sqrt{2 - r^2} \end{array} \right\}.$$

$$\Rightarrow \boxed{V = \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^1 \int_r^{2-\sqrt{2-r^2}} r \, dz \, dr \, d\theta.}$$

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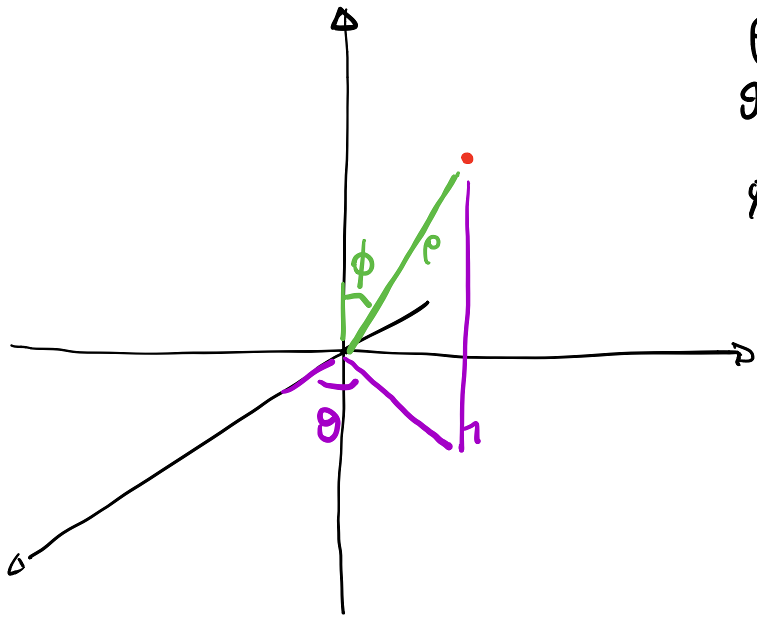
§12.7: $\iiint$ in spherical coordinates

__cylindrical__ coordinates: use those when integrand involves x^2+y^2 and/or rotational symmetry about z-axis

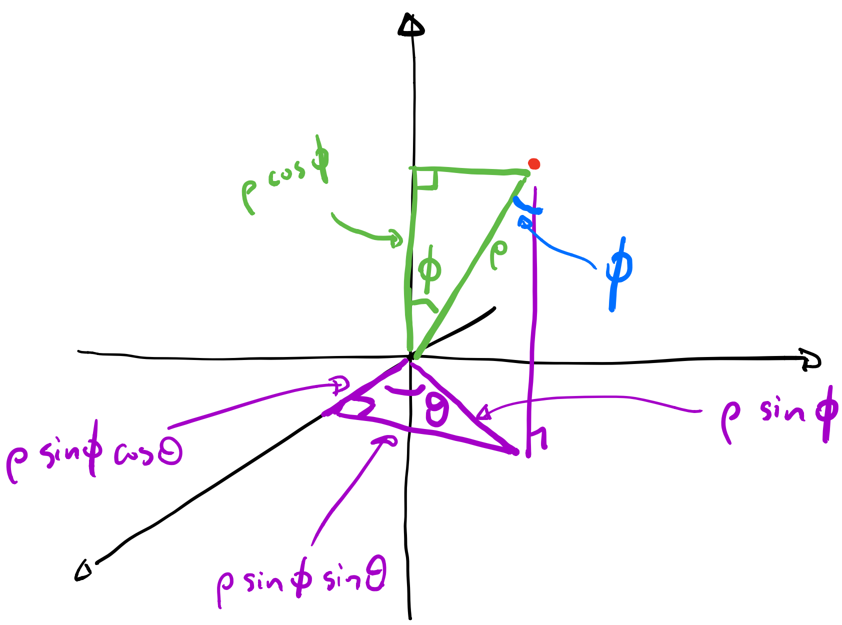
__spherical__ coordinates: use when integrand involves $x^2+y^2+z^2$, and/or rotational symmetry about origin.



ρ : distance to origin, $\in [0, \infty)$

θ : "longitude", $\in [0, 2\pi]$

ϕ : "latitude", $\in [0, \pi]$.



$$\boxed{\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}}$$

(note: $x^2+y^2+z^2 = \rho^2$)

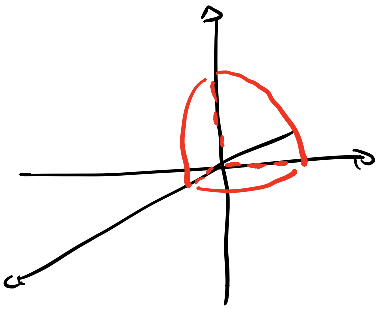
To convert $\iiint$ to spherical coords:

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

Fall '13, #5. $I = \iiint_E z^6 \, dV$, $E = \left\{ \begin{array}{l} x^2+y^2+z^2 \le 4 \\ x \ge 0,\ y \ge 0,\ z \ge 0 \end{array} \right\}$.

(a) Rewrite I in spherical coordinates.

(b) Compute I.



(a) $x^2+y^2+z^2 \le 4 \iff \rho^2 \le 4 \iff \boxed{0 \le \rho \le 2.}$

$x \ge 0 \iff \underbrace{\rho}_{\ge 0} \underbrace{\sin\phi}_{\ge 0} \cos\theta \ge 0 \iff \cos\theta \ge 0 \iff -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$

$y \ge 0 \iff \rho \sin\phi \sin\theta \ge 0 \iff \sin\theta \ge 0 \iff 0 \le \theta \le \pi$

$\iff 0 \le \theta \le \frac{\pi}{2}$

$z \ge 0 \iff \rho \cos\phi \ge 0 \iff \cos\phi \ge 0 \iff \boxed{0 \le \phi \le \pi/2.}$

$$\leadsto I = \int_0^2 \int_0^{\pi/2} \int_0^{\pi/2} (\rho \cos\phi)^6 \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^2 \int_0^{\pi/2} \int_0^{\pi/2} \rho^8 \sin\phi \cos^6\phi \, d\phi \, d\theta \, d\rho$$

$u = \cos\phi \Rightarrow du = -\sin\phi \, d\phi$

$\Rightarrow \sin\phi \, d\phi = -du$

$$= \int_0^2 \int_0^{\pi/2} \int_0^1 \rho^8 u^6 \, du \, d\theta \, d\rho$$

$$= \int_0^2 \int_0^{\pi/2} \left[\frac{1}{7} \rho^8 u^7 \right]_{u=0}^{1} d\theta \, d\rho$$

$$= \int_0^2 \int_0^{\pi/2} \frac{1}{7} \rho^8 \, d\theta \, d\rho$$

$$= \int_0^2 \frac{\pi}{14} \rho^8 \, d\rho$$

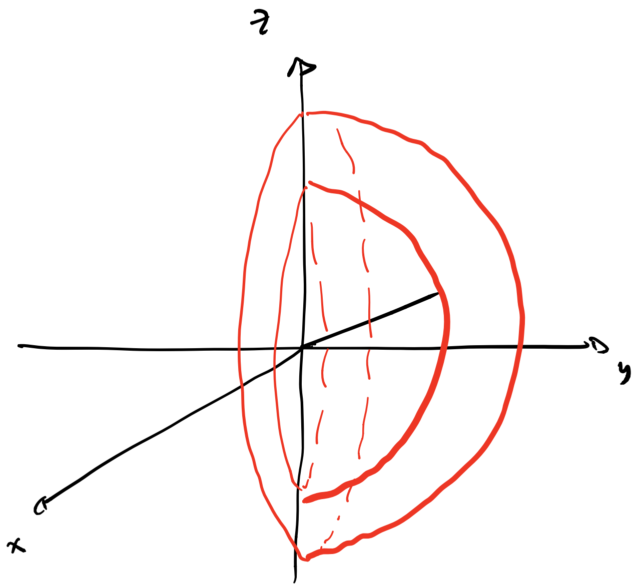
$$= \left[\frac{\pi}{126} \rho^9 \right]_0^2 = \frac{256}{63} \pi .$$

$\triangle$

<u>Spring '12, #7.</u> Use spherical coords to compute $I = \iiint_E x^2 \, dV$,

E bounded by $y = \sqrt{9 - x^2 - z^2}$, $y = \sqrt{16 - x^2 - z^2}$, xz-plane.

$\Rightarrow x^2 + y^2 + z^2 = 9$ $\quad\quad$ $\Rightarrow x^2 + y^2 + z^2 = 16$



$$E = \left\{ \begin{array}{c} 3 \le \rho \le 4 \\ 0 \le \phi \le \pi \\ 0 \le \theta \le \pi \end{array} \right\}.$$

Fall '12, #8. Calculate volume below $\{x^2+y^2+(z-1)^2=1\}$, above conical surface $z=\sqrt{x^2+y^2}$.

$\Rightarrow (z-1)^2 = 1-x^2-y^2$

$\Rightarrow z = 1 \pm \sqrt{1-x^2-y^2}$.

a. Intersection?

$$\sqrt{x^2+y^2} = 1 \pm \sqrt{1-x^2-y^2}$$

$$\Rightarrow x^2+y^2 = 1 \pm 2\sqrt{1-x^2-y^2} + 1 - x^2 - y^2$$

$$\Rightarrow 2x^2+2y^2 = 2 \pm 2\sqrt{1-x^2-y^2}$$

$$\Rightarrow x^2+y^2 = 1 \pm \sqrt{1-x^2-y^2}$$

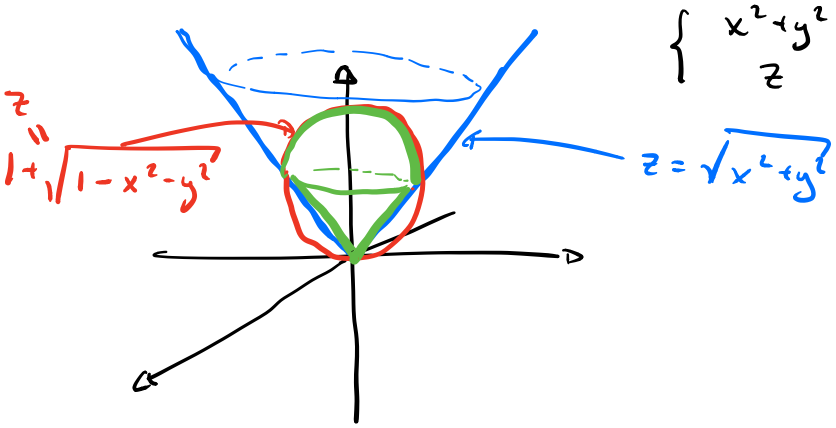
$$\Rightarrow x^2+y^2-1 = \pm\sqrt{1-x^2-y^2}$$

$$\Rightarrow (x^2+y^2)^2 - 2(x^2+y^2)+1 = 1-x^2-y^2$$

$$\Rightarrow (x^2+y^2)^2 - (x^2+y^2) = 0$$

$$\Rightarrow (x^2+y^2)(x^2+y^2-1) = 0 \rightarrow \begin{array}{c} x^2+y^2=0 \\ \underline{\text{or}} \\ x^2+y^2=1. \end{array}$$

$\Rightarrow$ intersection is $(0,0,0)$ and $\left\{ \begin{array}{l} x^2+y^2=1, \\ z=1 \end{array} \right\}$.



$$V = \iint_{\{x^2+y^2\le 1\}} \left(1 + \sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dA$$

$1 + \sqrt{1-r^2} - r$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$$= \int_0^{2\pi} \int_0^1 \left(r + r\sqrt{1-r^2} - r^2\right) dr \, d\theta$$

$$= \int_0^{2\pi} \left(\int_0^1 (r - r^2)\, dr + \int_0^1 r\sqrt{1-r^2}\, dr \right) d\theta$$

$u = 1 - r^2$

$\Rightarrow du = -2r\,dr$

$\Rightarrow r\,dr = -\frac{1}{2} du$

$$= \int_0^{2\pi} \left(\int_0^1 (r - r^2)\, dr - \frac{1}{2}\int_1^0 \sqrt{u}\, du \right) d\theta$$

$$= \int_0^{2\pi} \left(\left[\frac{1}{2}r^2 - \frac{1}{3}r^3\right]_{r=0}^{r=1} - \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=0} \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{6} - \left(0 - \frac{1}{6}\right) \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}\, d\theta = \frac{2\pi}{3}.$$

$\triangle$.

§13.1 : Vector fields.

Definition. Given a region $U \subset \mathbb{R}^2$, a vector field is a function that assigns to each point of U a 2D vector.

$$\underline{F}(x,y) = \langle P(x,y), Q(x,y) \rangle.$$

Similarly for a vector field on a region in $\mathbb{R}^3$, but it will spit out 3D vectors.

Ex 1. $\underline{F}(x,y) := \langle -y, x \rangle$, on $\mathbb{R}^2$. Draw it.

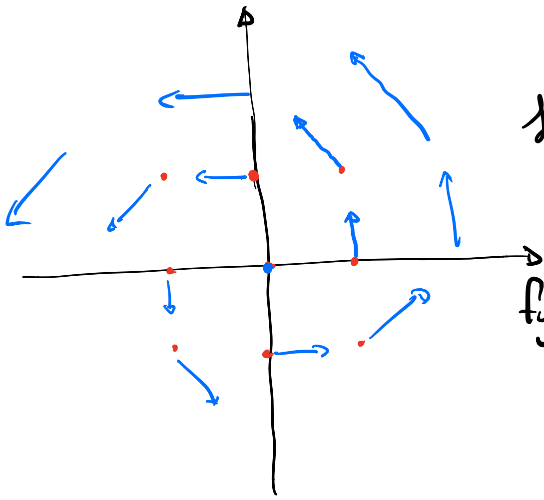
$F(0,0) = \langle 0,0 \rangle$

$F(1,0) = \langle 0,1 \rangle$

$F(1,1) = \langle -1,1 \rangle$

$F(0,1) = \langle -1,0 \rangle$

$\vdots$



Is this conservative?

Say $\langle -y, x \rangle = \nabla f = \langle f_x, f_y \rangle$.

$f_x = -y \implies f = -xy + \alpha(y)$.

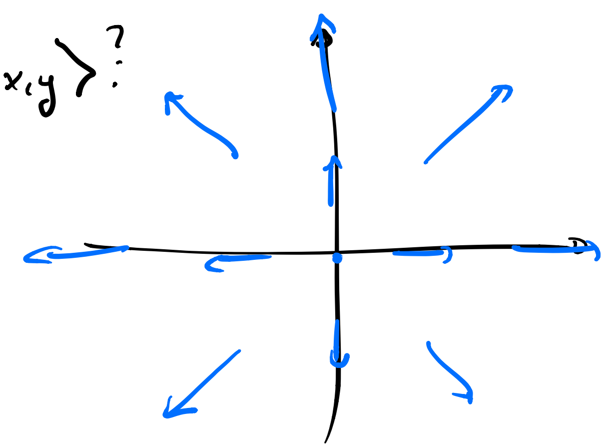
$f_y = x \implies (-xy + \alpha_y)_y = x$

$\implies -x + \alpha'(y) = x$

$\implies \alpha'(y) = 2x$

$\implies \langle -y, x \rangle$ not conservative!

$\underline{G}(x,y) := \langle x, y \rangle$?



Gradient vector fields.

Given any $f(x,y)$. $\nabla f(x,y)$ is a 2D vector field

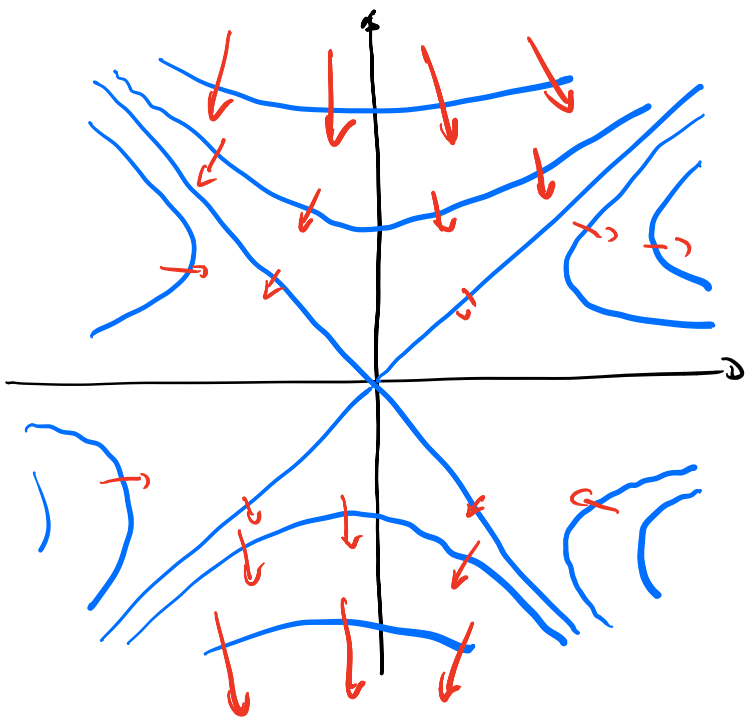
$$= \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle.$$

Given any $g(x,y,z)$, $\nabla g(x,y,z)$ is a 3D vector fields.

Recall that ∇f is perpendicular to the level surfaces / curves of f.

Eg., $f(x,y) = x^2y - y^3 \rightsquigarrow \nabla f = \langle 2xy, x^2 - 3y^2 \rangle$.

level curves are $x^2y - y^3 = c$



a vector field $\underline{F}$ is conservative if there exists some f with $\underline{F} = \nabla f$.

"potential function".

Ex. Say we have two objects w/ masses m, M.

@ $\underline{x}$ @ $(0,0,0)$

Then the gravitational force on the object @ $\underline{x}$ is:

$$\boxed{\underline{F}(\underline{x}) := -\frac{mMG}{|\underline{x}|^3} \cdot \underline{x}}$$

"gravitational vector field".

Note: $\underline{F} = \nabla f$, $\boxed{f := \frac{mMG}{|\underline{x}|}}$.

OH: W 1-2:30, F 2-3
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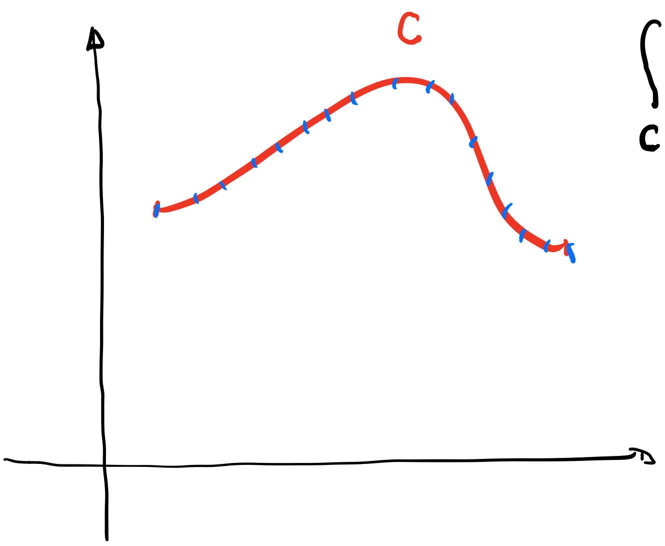
§13.2: Line integrals.

say have body in $\mathbb{R}^3$, occupying region E, and density given by $\delta(x,y,z)$. $\rightsquigarrow$ mass of body $= \iiint_E \delta(x,y,z)\, dV$.

$$\int_C f(x,y)\, ds \quad \text{"mass"}$$

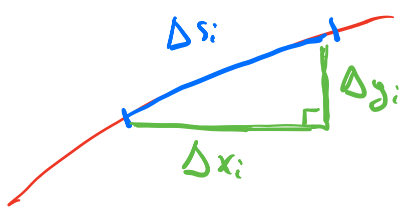
$$\int_C \underline{F}(x,y,z)\cdot d\underline{r} \quad \text{"work"}.$$



$$\int_C f(x,y)\, ds \overset{(*)}{:=} \lim_{\max \Delta s_i} \sum f(x_i^*, y_i^*)\, \Delta s_i$$

length of i-th segment

$$\approx f(\underline{r}(t_i^*))\sqrt{x'(t_i^*)^2 + y'(t_i^*)^2}\, \Delta t_i$$
$$= f(\underline{r}(t_i^*))\, |\underline{r}'(t_i)|\, \Delta t_i$$



$$\Delta s_i \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

How to actually compute?

Say $\underline{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$ is a parametrization of our curve.

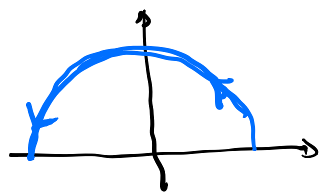
then: $$\Delta s_i \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \approx \sqrt{(x'(t_i^*)\Delta t_i)^2 + (y'(t_i)\Delta t_i)^2}$$
$$= \sqrt{(x')^2 + (y')^2}\, \Delta t_i$$

$$\rightsquigarrow \quad \boxed{\int_C f(x,y)\, ds = \int_a^b f(\underline{r}(t))\, |\underline{r}'(t)|\, dt.}$$

Similarly:

$$\boxed{\begin{aligned} \int_C f(x,y)\, dx &= \int_a^b f(\underline{r}(t))\, x'(t)\, dt \\ \int_C f(x,y)\, dy &= \int_a^b f(\underline{r}(t))\, y'(t)\, dt. \end{aligned}}$$

Ex 1. Compute $I = \int_C (2 + x^2 y)\, ds$, C the top half of the unit circle.



a. $\underline{r}(t) := \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$.

$|\underline{r}'(t)| = |\langle -\sin t, \cos t \rangle| = 1.$

$$I = \int_0^\pi (2 + \cos^2 t \sin t) \cdot 1 \, dt$$

$$= \left[2t - \frac{1}{3} \cos^3 t \right]_{t=0}^{t=\pi} = \left(2\pi - \frac{1}{3} \cdot (-1)^3\right) - \left(0 - \frac{1}{3} \cdot 1^3\right)$$

$$= 2\pi + \frac{2}{3}.$$

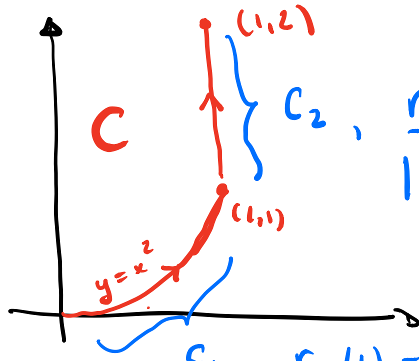
$\triangle$

(Note: $\int_{-C} f(x,y)\, ds = \int_C f(x,y)\, ds.$

$\int_{-C} f(x,y)\, dx = -\int_C f(x,y)\, dx$

$\int_{-C} f(x,y)\, dy = -\int_C f(x,y)\, dy$)

$\underline{Ex\ 2}$. $I = \int_C 2x\, ds$,



C_2, $\underline{r}_2(t) := \langle 1, t\rangle$, $1 \le t \le 2$,
$|\underline{r}_2'(t)| = 1$

C_1, $\underline{r}_1(t) = \langle t, t^2\rangle$, $0 \le t \le 1$.
$|\underline{r}'(t)| = |\langle 1, 2t\rangle| = \sqrt{1+4t^2}$

$$I = \int_{C_1} 2x\, ds + \int_{C_2} 2x\, ds$$

$$= \int_0^1 2t\sqrt{1+4t^2}\, dt + \int_1^2 2\, dt$$

$u = 1+4t^2$
$du = 8t\, dt$
$2t\, dt = \frac{1}{4}\, du$

$$= \int_1^5 \frac{1}{4}\sqrt{u}\, du + 2 = \left[\frac{1}{4}\cdot\frac{2}{3}\cdot u^{3/2}\right]_{u=1}^{u=5} + 2$$

$$= \frac{5\sqrt{5}-1}{6} + 2.$$

Δ

$\underline{\text{Line integrals in space}}$.

$\underline{Ex\ 5}$. $I = \int_C y \sin z\, ds$, C circular helix, $\underline{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$.

$\underline{a}$. $|\underline{r}'(t)| = |\langle -\sin t, \cos t, 1\rangle| = \sqrt{2}$.

$$I = \int_0^{2\pi} \sin t \cdot \sin t \cdot \sqrt{2}\, dt = \int_0^{2\pi} \frac{\sqrt{2}}{2}(1-\cos 2t)\, dt$$

$$= \left[\frac{\sqrt{2}}{2}\cdot\left(t - \frac{1}{2}\sin 2t\right)\right]_{t=0}^{t=2\pi}$$

$$= \sqrt{2}\,\pi.$$

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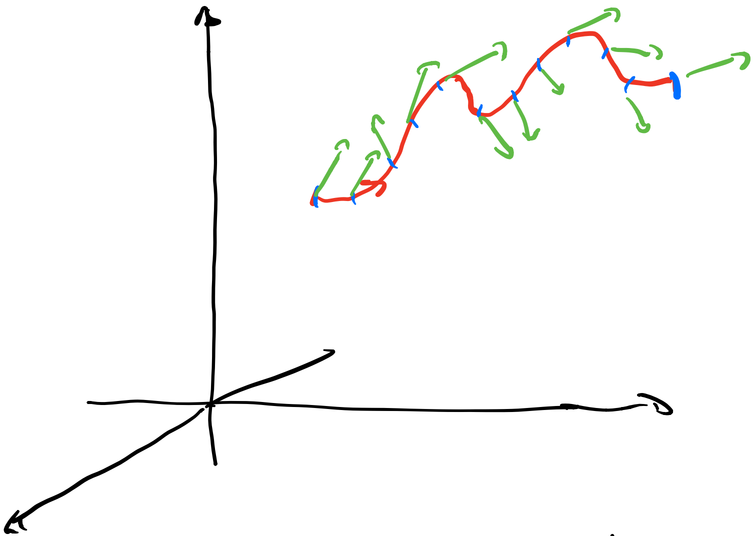
Line integrals of vector fields.



$W \approx \sum_i \underline{F}(x_i^*, y_i^*, z_i^*) \cdot \underline{T}(x_i^*, y_i^*, z_i^*) \cdot \Delta s_i$

unit tangent vector

$\underline{T} = \dfrac{\underline{r}'}{|\underline{r}'|}$

$\Rightarrow \underline{T} \cdot \Delta s_i = \dfrac{\underline{r}'}{|\underline{r}'|} \cdot |\underline{r}'| \, dt$

$= \underline{r}' \, dt$

$$\rightsquigarrow W = \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t) \, dt$$

(note, $\int_{-C} \underline{F} \cdot d\underline{r} = -\int_C \underline{F} \cdot d\underline{r}$)

$$\boxed{\int_C \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t) \, dt}.$$

Ex 7. Work done by $\underline{F} = \langle x^2, -xy \rangle$ as particle moves along $\langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$.

§13.3: Fundamental theorem of line integrals.

Recall $\boxed{\text{FTC: } \displaystyle\int_a^b f'(x)\,dx = f(b) - f(a).}$

$\boxed{\begin{array}{l}\textbf{Fundamental theorem of line integrals: Say } C \text{ a curve} \\ \text{from } p_1 \text{ to } p_2. \text{ Then:} \\ \displaystyle\int_C \nabla f \cdot d\underline{r} = f(p_2) - f(p_1).\end{array}}$

<u>Proof in 3D</u>. Choose $\underline{r}$ a parametrization of our curve.

$\langle x(t), y(t), z(t) \rangle$

$$\int_C \nabla f \cdot d\underline{r} = \int_a^b \nabla f(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt}\Big(f(\underline{r}(t)) \Big)\, dt$$

$$\overset{\text{FTC}}{=} f(\underline{r}(b)) - f(\underline{r}(a))$$

$$= f(p_2) - f(p_1).$$

$\blacksquare$

Important consequence of FTL I: If $\underline{F}$ is a conservative vector field, then $\int_C \underline{F} \cdot d\underline{r}$ depends only on endpoints of C, not on which path you take to get from p_1 to p_2.

a criterion for conservativity

on $D \subset \mathbb{R}^2$

Thm. a. If $\underline{F} = \langle P(x,y), Q(x,y) \rangle$ is conservative, then:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

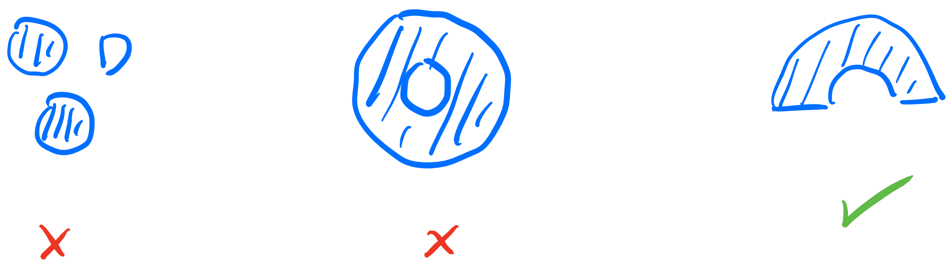
b. If domain D is open and simply-connected, and if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\underline{F} = \langle P, Q \rangle$ is conservative.

Proof of a. Say $\underline{F}$ is conservative, then exists $f(x,y)$ w/ $\underline{F} = \nabla f = \langle \underbrace{\frac{\partial f}{\partial x}}_{P}, \underbrace{\frac{\partial f}{\partial y}}_{Q} \rangle$.

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial x}.$$

D simply-connected means:

* D is "one piece"
* D has no holes



Ex 2. Is $\underline{F} = \langle x-y,\ x-2 \rangle$ conservative?

Domain is $\mathbb{R}^2$ ✓

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x-2) = 1$$

$$\neq$$

$\Longrightarrow$ not conservative! $\triangle$

Ex 3. Same Q, $\underline{F} = \langle 3+2xy,\ x^2-3y^2 \rangle$.

domain $= \mathbb{R}^2$. ✓

$$\frac{\partial P}{\partial y} = 2x, \qquad \frac{\partial Q}{\partial x} = 2x$$

$=$ $\Longrightarrow$ F conservative!

What's an f w/ $\underline{F} = \nabla f$?

$$\begin{cases} f_x = 3 + 2xy & (1) \\ f_y = x^2 - 3y^2 & (2) \end{cases}$$

$$(1) \Rightarrow f = \int (3 + 2xy)\, dx$$

$$= 3x + x^2 y + \alpha(y).$$

$$(2) \Rightarrow \left(3x + x^2 y + \alpha(y)\right)_y = x^2 - 3y^2$$

$$\Rightarrow \cancel{x^2} + \alpha'(y) = \cancel{x^2} - 3y^2$$

$$\Rightarrow \alpha'(y) = -3y^2 \Rightarrow \alpha(y) = -y^3 + C.$$

$$\Rightarrow \boxed{f = 3x + x^2 y - y^3 + C.}$$

$\triangle$

Spring '15, #6. $\underline{F} = \langle e^x \cos y,\ 2y - e^x \sin y \rangle$.

a. Find f w/ $\underline{F} = \nabla f = \langle f_x, f_y \rangle$.

b. Evaluate $I = \int_C \underline{F} \cdot d\underline{r}$, C parametrized by

$$\underline{r}(t) = \langle t + \sin \frac{\pi t}{2},\ t + \cos \frac{\pi t}{2} \rangle, \quad 0 \le t \le 1.$$

a. $$\begin{cases} f_x = e^x \cos y & (1) \\ f_y = 2y - e^x \sin y & (2) \end{cases}$$

(2) $\Rightarrow$ $$f = \int (2y - e^x \sin y)\, dy = y^2 + e^x \cos y + \alpha(x)$$

(1) $\Rightarrow$ $$\left(y^2 + e^x \cos y + \alpha(x) \right)_x = e^x \cos y$$

$$\Rightarrow \cancel{e^x \cos y} + \alpha'(x) = \cancel{e^x \cos y}$$

$$\Rightarrow \alpha'(x) = 0 \Rightarrow \alpha = C.$$

$\Rightarrow$ can take $f := y^2 + e^x \cos y$.

b. Evaluate $I = \int_C \underline{F} \cdot d\underline{r}$, C parametrized by

$$\underline{r}(t) = \left\langle t + \sin \frac{\pi t}{2},\ t + \cos \frac{\pi t}{2} \right\rangle,$$

$0 \le t \le 1$.

$$I = \int_C \underline{F} \cdot d\underline{r} = \int_C \nabla f \cdot d\underline{r}$$

$$\overset{FTLI}{=} f(\underline{r}(1)) - f(\underline{r}(0))$$

$$= f(2,1) - f(0,1)$$

$\triangle$

theorem	quantity 1	quantity 2
FTC	$\int_a^b f'(x)\,dx$	$f(b) - f(a)$
FTLI	$\int_C \nabla f \cdot d\mathbf{r}$	$f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
Green's theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$	$\int_C (P\,dx + Q\,dy)$.

C parametrized by $\mathbf{r}$, $a \le t \le b$

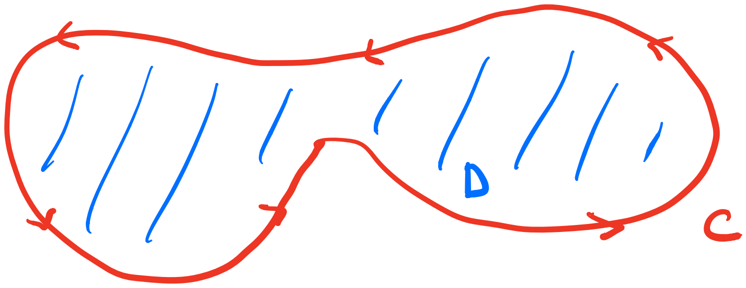
today.

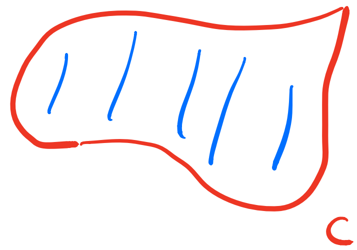
go around it counter-clockwise.

Green's theorem. Let C be a positively-oriented, piecewise-smooth, simple closed curve in the plane. Let D be the region it bounds. Let $P(x,y)$, $Q(x,y)$ be defined on D. Then:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C (P\,dx + Q\,dy).$$

$\left(= \int_C \mathbf{F} \cdot d\mathbf{r},\ \mathbf{F} = \langle P, Q \rangle \right)$





BTW, what is Green's theorem saying when $\langle P, Q \rangle$ is conservative?

- $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$? Well, if $\langle P, Q \rangle$ is conservative,

then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \implies \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D 0 \, dA = 0.$

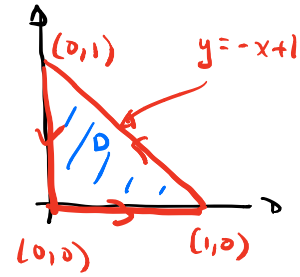
- $\int_C \underline{F} \cdot d\underline{r}$, $\underline{F} := \langle P, Q \rangle$? Parametrize C by $\underline{r}(t)$, $a \le t \le b$.

Then $\underline{r}(a) = \underline{r}(b)$. So:

$$\int_C \underline{F} \cdot d\underline{r} = \int_C \nabla f \cdot d\underline{r} \overset{\text{FTLI}}{=} f(\underline{r}(b)) - f(\underline{r}(a)) = 0.$$

$\triangle$

Ex 1. Evaluate $I = \int_C \left(\underbrace{x^4}_{P}\, dx + \underbrace{xy}_{Q}\, dy\right)$, $C =$



a. Note, $D = \left\{ \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le -x+1 \end{array} \right\}$.

So,

$$I = \int_C \left(x^4\, dx + xy\, dy \right)$$

$$\overset{G}{=} \iint_D \left((xy)_x - (x^4)_y \right) dA$$

$$= \int_0^1 \int_0^{-x+1} y \, dy \, dx$$

$$= \int_0^1 \left[\tfrac{1}{2} y^2 \right]_0^{-x+1} dx$$

$$= \int_0^1 \tfrac{1}{2} (x-1)^2 \, dx$$

$$= \left[\tfrac{1}{6} (x-1)^3 \right]_0^1 = \tfrac{1}{6} \cdot 0 - \tfrac{1}{6} \cdot (-1)$$

$$= 1/6.$$

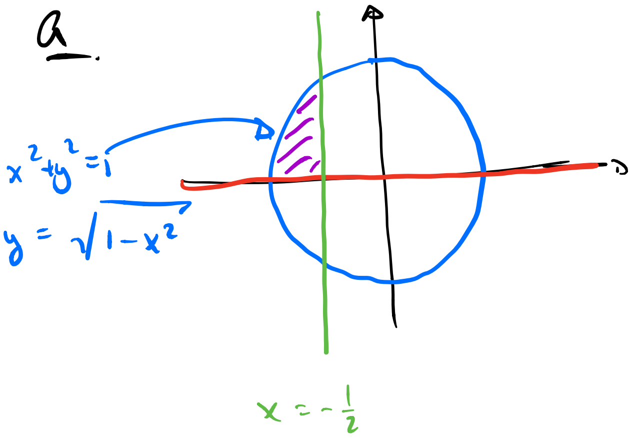
Δ

Spring '16, #6. Use Green's theorem to compute

$$I = \int_C \underline{F} \cdot d\underline{r}, \qquad \underline{F} = \langle \underbrace{-\tfrac{1}{2} x^2 y^2}_{P}, \underbrace{x y^3}_{Q} \rangle, \quad C$$

the boundary of the region D lying inside $x^2+y^2=1$, above x-axis, to the left of $x=-\frac{1}{2}$, w/ counterclockwise orientation.

a.



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (xy^3)_x - (-\tfrac{1}{2}x^2y^2)_y$$

$$= y^3 - (-x^2y)$$

$$= x^2y + y^3.$$

$$I = \int_C \underline{F} \cdot d\underline{r} \overset{G}{=} \iint_D (x^2y+y^3)\, dA$$

$$= \int_{-1}^{-1/2} \int_0^{\sqrt{1-x^2}} (x^2y+y^3)\, dy\, dx$$

$$= \int_{-1}^{-1/2} \left[\tfrac{1}{2} x^2 y^2 + \tfrac{1}{4} y^4 \right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{-1/2} \underbrace{\left(\tfrac{1}{2} \cdot x^2 \cdot (1-x^2) + \tfrac{1}{4}(1-x^2)^2 \right)}_{\cancel{\frac{1}{2}x^2} - \frac{1}{2}x^4 + \frac{1}{4}(1 \cancel{-2x^2} + x^4) \,=\, \frac{1}{4} - \frac{1}{4}x^4} dx$$

$$= \int_{-1}^{-1/2} \left(\frac{1}{4} - \frac{1}{4} x^4 \right) dx$$

$$= \left[\frac{1}{4} x - \frac{1}{20} x^5 \right]_{-1}^{-1/2} .$$

Δ

§13 final problems

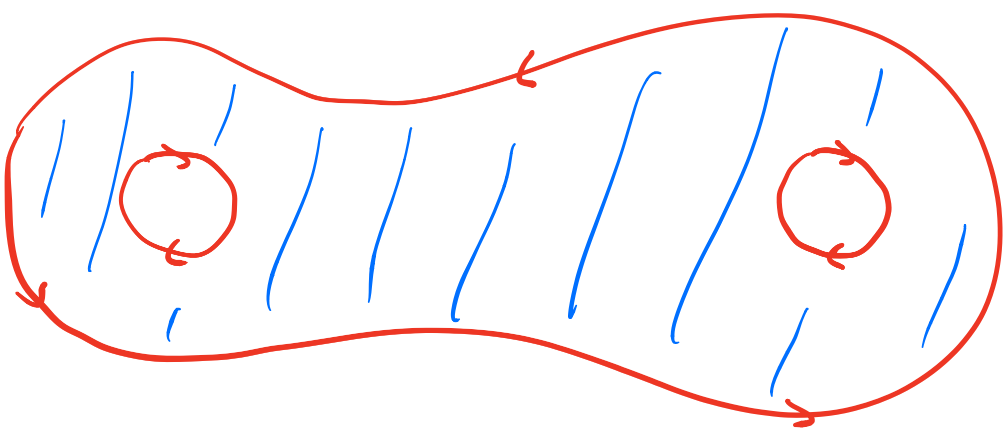
$\iiint$ in spherical coords w/ complicated bdry

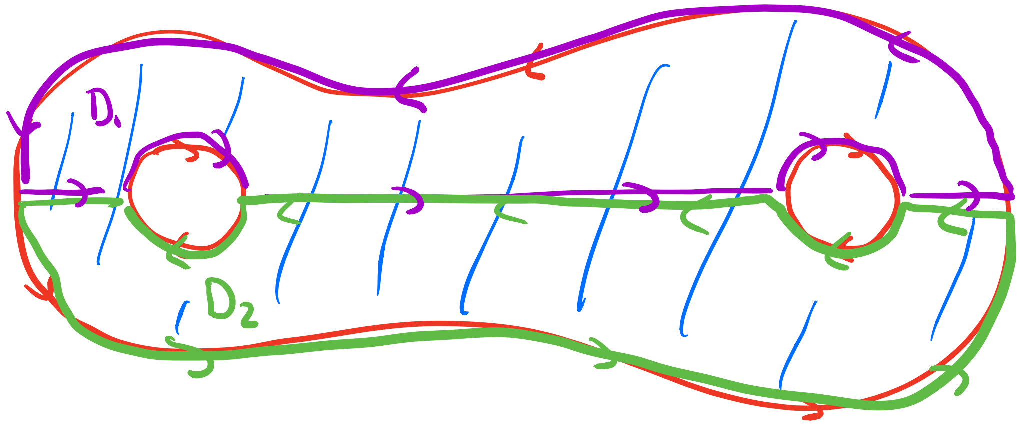
absolute max/min

nasty Lag multipliers

<u>An extension of Green's theorem</u>: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C (P dx + Q dy)$

even when D has holes, where $\int_C$ includes integrals over bdries of holes, oriented clockwise.





D1
D2