§13.5: Curl, divergence.

name	input	output	in symbols.
gradient	function	vector field	∇f
curl	3D vector field	3D vector field	$\nabla \times \underline{F}$
divergence	vector field	function	$\nabla \cdot \underline{F}$

Curl.

Def. The curl of $\underline{F} = \langle P, Q, R \rangle$ is the following v.f:

$$\operatorname{curl} \underline{F} := \nabla \times \underline{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$= \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

looks like the thing that must vanish for a 2D vector field to be conservative

Ex 1. $\operatorname{curl}\langle xz, xyz, -y^2\rangle$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \langle -2y - xy,\ x - 0,\ yz - 0\rangle$$

$$= \langle -2y - xy,\ x,\ yz\rangle.$$

$\triangle$

Thm. (1) If $\underline{F}$ conservative, then $\operatorname{curl} \underline{F} = \underline{0}$.

(2) If $\underline{F}$ defined on $\mathbb{R}^3$, and $\operatorname{curl} \underline{F} = 0$, then $\underline{F}$ conservative.

#6 from... some final. $\underline{F} := \langle 2x\cos y - 2z^3,\ 3 + 2ye^z - x^2\sin y,\ y^2e^z - 6xz^2\rangle$.

(a) Conservative? If so, find f w/ $\underline{F} = \nabla f$.

(b) Evaluate $\int_C \underline{F}\cdot d\underline{r}$, C parametrized by $\underline{r}(t) := \langle t+1,\ 4t - 4t^2,\ t^3 - t\rangle$.

a. (1)

$$\operatorname{curl} \underline{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2x\cos y - 2z^3 & 3 + 2ye^z - x^2\sin y & y^2e^z - 6xz^2 \end{vmatrix}$$

$$= \langle 2ye^z - 2ye^z,\ -6z^2 - (-6z^2),\ -2x\sin y - (-2x\sin y)\rangle$$

$$= \langle 0, 0, 0\rangle = \underline{0}.$$

$\operatorname{curl} \underline{F} = \underline{0}$, domain $\mathbb{R}^3 \implies \underline{F}$ conservative.

f w/ $\underline{F} = \nabla f$?

$$\begin{cases} f_x = 2x\cos y - 2z^3 & (1) \\ f_y = 3 + 2ye^z - x^2 \sin y & (2) \\ f_z = y^2 e^z - 6xz^2 & (3) \end{cases}$$

$(1) \implies f = \int (2x\cos y - 2z^3)\, dx = x^2 \cos y - 2xz^3 + \alpha(y,z)$

$(2) \implies \left(x^2\cos y - 2xz^3 + \alpha(y,z)\right)_y = 3 + 2ye^z - x^2 \sin y$

$\implies -\cancel{x^2 \sin y} + \alpha_y = 3 + 2ye^z - \cancel{x^2 \sin y}$

$\implies \alpha_y = 3 + 2ye^z$

$\implies \alpha = \int (3 + 2ye^z)\, dy$

$= 3y + y^2 e^z + \beta(z)$.

(3) $f_z = y^2 e^z - 6xz^2 \implies \left(x^2 \cos y - 2xz^3 + 3y + y^2 e^z + \beta(z)\right)_z$

$= y^2 e^z - 6xz^2$

$\implies \cancel{-6xz^2} + \cancel{y^2 e^z} + \beta' = \cancel{y^2 e^z} - \cancel{6xz^2}$

$\implies \beta' = 0 \implies \beta = C$.

$$\Rightarrow \boxed{f = x^2\cos y - 2xz^3 + 3y + y^2 e^z} \, .$$

(b) Evaluate $\overset{I:}{\int_C} \underline{F} \cdot d\underline{r}$, C parametrized by $\underline{r}(t) := \langle t+1,\ 4t-4t^2,\ t^3-t \rangle$, $0 \le t \le 1$.

$$I = \int_C \nabla f \cdot d\underline{r} \overset{\text{FTLI}}{=} f(\underline{r}(1)) - f(\underline{r}(0))$$

$$= f(2,0,0) - f(1,0,0)$$

$$= (4 - 0 + 0 + 0) - (1 - 0 + 0 + 0)$$

$$= 3.$$

Δ

Divergence.

Def. The divergence is defined as:

$$\operatorname{div} \langle P, Q \rangle = \nabla \cdot \langle P, Q \rangle = P_x + Q_y$$

$$\operatorname{div} \langle P, Q, R \rangle = \nabla \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z \, .$$

Ex 4. $\operatorname{div} \langle xz, xyz, -y^2 \rangle = (xz)_x + (xyz)_y + (-y^2)_z$

$$= z + xz.$$

Δ

Theorem: $\operatorname{div} \operatorname{curl} \underline{F} = 0$.

Corollary: If $\operatorname{div} \underline{G} \neq 0$, then there's no v.f. $\underline{F}$ with $G = \operatorname{curl} \underline{F}$.

Thm. (1) If $\underline{F}$ conservative, then $\operatorname{curl} \underline{F} = \underline{0}$.

(2) If $\underline{F}$ defined on $\mathbb{R}^3$, and $\operatorname{curl} \underline{F} = 0$, then $\underline{F}$ conservative.

Pf of (1). $\underline{F} = \nabla f$.

$$\operatorname{curl} \underline{F} = \operatorname{curl} \nabla f = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

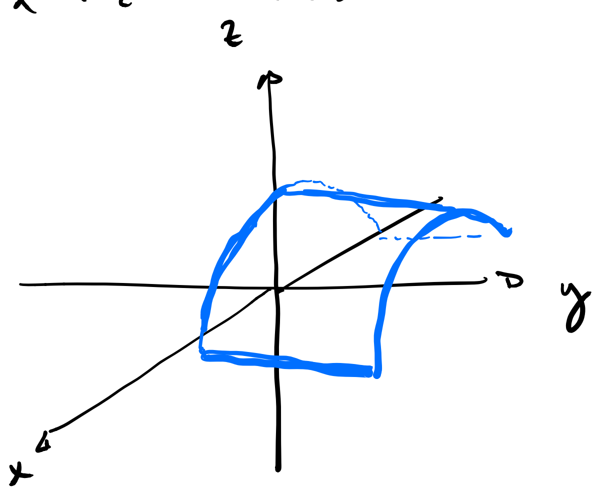
$$= \underline{0}. \qquad \triangle$$

§13.6: Parametric surfaces.

Ex 1. Consider $\underline{r}(u,v) := \langle \overset{x}{2\cos u}, \overset{y}{v}, \overset{z}{2\sin u} \rangle$,

$0 \leq u \leq \pi, \quad 0 \leq v \leq 1.$

Note $x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4.$



§ 13.6 : Parametric surfaces and their areas

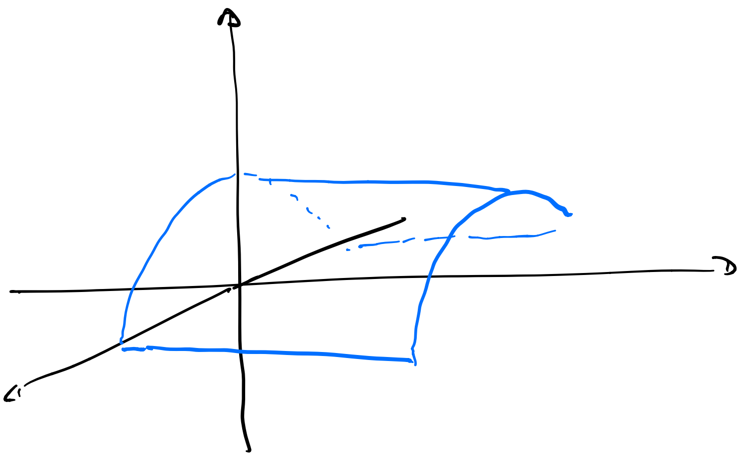
$\underline{r}(t)$: parametrizes curve (in $\mathbb{R}^2, \mathbb{R}^3$)

$\underline{r}(u,v)$: parametrizes surface (in $\mathbb{R}^3$)

Ex from last time : $\underline{r}(u,v) := \langle \overset{x(u,v)}{\cos u}, \overset{y}{v}, \overset{z}{\sin u} \rangle$,

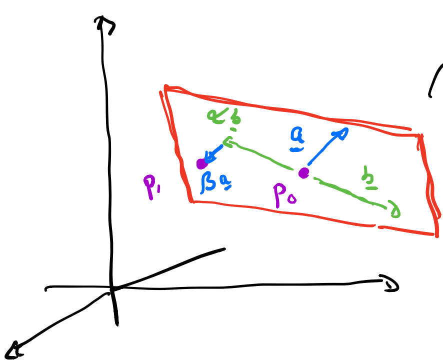
$0 \le u \le \pi, \quad 0 \le v \le 1.$

Note : $x^2 + z^2 = \cos^2 u + \sin^2 u = 1 \implies$ this is some chunk of cylinder $\{x^2 + z^2 = 1\}$.



$\triangle$

Ex 3 : How to parametrize plane P thru p_0, and containing vectors $\underline{a}, \underline{b}$ (assume $\underline{a}, \underline{b}$ not parallel)?



any point on P can be written as

$p_0 + ?\,\underline{a} + ?\,\underline{b}$

$\rightsquigarrow$ can parametrize P by $\underline{r}(u,v) := p_0 + u\underline{a} + v\underline{b}$,

$-\infty < u < \infty$

$-\infty < v < \infty$.

$\triangle$

Ex 4-ish. Parametrize top half of $\{x^2+y^2+z^2=25\}$.

$\Longleftrightarrow \rho^2 = 25$

$\Longleftrightarrow \rho = 5$.

A. Let's see if spherical coordinates help.

$$\rightsquigarrow \underline{r}(\theta, \phi) := \langle 5\sin\phi\cos\theta,\ 5\sin\phi\sin\theta,\ 5\cos\phi\rangle,$$

$$0 \le \theta \le 2\pi,$$

$$0 \le \phi \le \pi/2.$$

Δ

(spherical coords: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$)

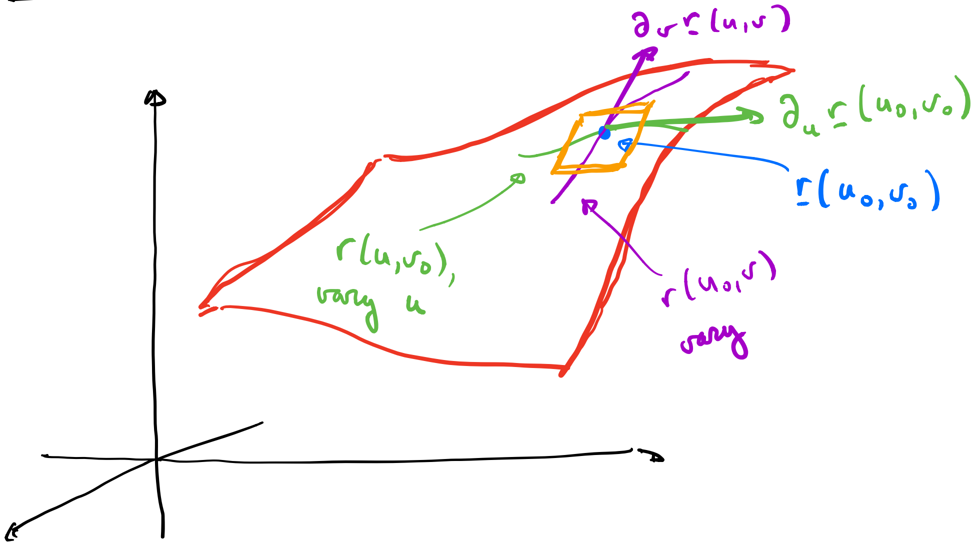
Ex. parametrize $S :=$ graph of $f(x,y)$, $a \le x \le b$, $c \le y \le d$.

A. $S = \left\{ z = f(x,y) \;\middle|\; \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array} \right\}$.

$$\underline{r}(u,v) := \langle u, v, f(u,v)\rangle, \quad \begin{array}{l} a \le u \le b \\ c \le v \le d. \end{array}$$

Δ

Tangent planes to parametrized surfaces.



$\rightsquigarrow$ can parametrize tangent plane by:

$$\boxed{\tilde{r}(a,b) := \underline{r}(u_0,v_0) + a\,\partial_u \underline{r}(u_0,v_0) + b\,\partial_v \underline{r}(u_0,v_0).}$$

or equivalently: normal vector is $\partial_u \underline{r}(u_0,v_0) \times \partial_v \underline{r}(u_0,v_0)$

$$\rightsquigarrow \boxed{\left(\partial_u \underline{r}(u_0,v_0) \times \partial_v \underline{r}(u_0,v_0)\right) \cdot \left(\langle x,y,z\rangle - r(u_0,v_0)\right) = 0.}$$

<u>Ex 8</u>. tangent plane to surface parametrized by $\underline{r}(u,v) = \langle u^2, v^2, u+2v\rangle$ @ $(1,1,3)$.

<u>Q</u>. What are u_0, v_0? $\begin{matrix} u_0^2 = 1 \\ v_0^2 = 1 \end{matrix} \Longrightarrow u_0 = \pm 1,\ v_0 = \pm 1$.

$u_0 + 2v_0 = 3 \Longrightarrow u_0 = v_0 = 1$.

$\partial_u \underline{r} = \langle 2u, 0, 1\rangle \overset{@(1,1)}{\rightsquigarrow} \langle 2,0,1\rangle$.

$\partial_v \underline{r} = \langle 0, 2v, 2\rangle \overset{@(1,1)}{\rightsquigarrow} \langle 0,2,2\rangle$.

$\Longrightarrow$ normal vector is $\begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \langle -2, -4, 4\rangle$.

$\Rightarrow$ tangent plane is $-2\cdot(x-1)+(-4)\cdot(y-1)+4(z-3)=0$

$\Rightarrow -2x-4y+4z = -2-4+12=6$

$\Rightarrow \boxed{-x-2y+2z=3.}$ Δ

Surface area.

$$\boxed{\begin{array}{l}\text{Surface area of surface parametrized by } \underline{r}(u,v),\ (u,v)\in D: \\[1em] A = \displaystyle\iint_D |\partial_u \underline{r} \times \partial_v \underline{r}|\, dA.\end{array}}$$

Spring '14, #8. Compute area of surface parametrized by

$$\underline{r}(u,v) := \langle u, uv, u^2\rangle, \quad \begin{array}{l}0\le u\le 1,\\ 0\le v\le 1.\end{array}$$

a. $\underline{r}_u = \langle 1, v, 2u\rangle$, $\underline{r}_v = \langle 0, u, 0\rangle$

$$\Rightarrow \underline{r}_u \times \underline{r}_v = \begin{vmatrix} i & j & k \\ 1 & v & 2u \\ 0 & u & 0 \end{vmatrix} \Rightarrow \langle -2u^2, 0, u\rangle$$

$$\Rightarrow |\underline{r}_u \times \underline{r}_v| = \sqrt{4u^4+u^2}.$$

$$A = \int_0^1\int_0^1 \sqrt{4u^4+u^2}\, du\, dv = \int_0^1\int_0^1 u\cdot\sqrt{4u^2+1}\, du\, dv$$

$w = 4u^2+1$

$\Rightarrow dw = 8u\, du \Rightarrow u\, du = \frac{1}{8}\, dw$

$$= \int_0^1\int_1^5 \frac{1}{8}\sqrt{w}\, dw\, dv$$

$$= \int_0^1 \left[\frac{1}{8} \cdot \frac{2}{3} \cdot w^{3/2} \right]_{w=1}^{w=5} dv$$

$$= \int_0^1 \frac{1}{12} \cdot \left(5\sqrt{5} - 1 \right) dv$$

$$= \boxed{\frac{1}{12} \left(5\sqrt{5} - 1 \right).}$$

Δ

$\{z = f(x,y)\}$

Surface area of graph of $f(x,y)$, $(x,y) \in D$?

$$\underline{r}(u,v) := \langle u, v, f(u,v) \rangle, \quad (u,v) \in D.$$

$$\Longrightarrow \underline{r}_u = \langle 1, 0, f_u \rangle, \quad \underline{r}_v = \langle 0, 1, f_v \rangle$$

$$\Longrightarrow \underline{r}_u \times \underline{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = \langle -f_u, -f_v, 1 \rangle$$

$$\Longrightarrow |\underline{r}_u \times \underline{r}_v| = \sqrt{1 + f_u^2 + f_v^2}$$

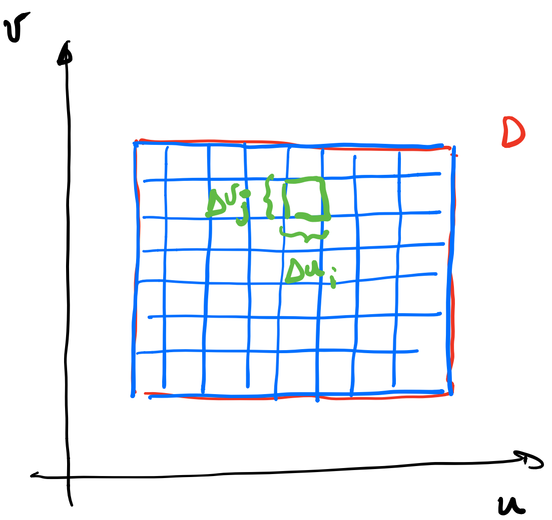
$$\Longrightarrow \boxed{A = \iint_D \sqrt{1 + f_u^2 + f_v^2} \, dA.}$$

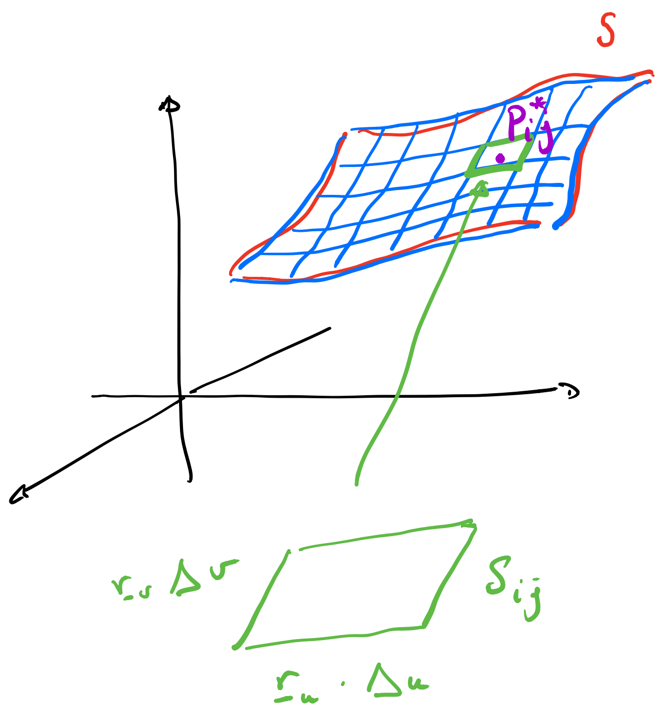
§13.7: Surface integrals.

$\iint_S f \, dS$ over parametrized surfaces.

think: mass of surface w/ varying density.

take S, parametrized by $\underline{r}(u,v)$, $(u,v) \in D$.





area $= |\underline{r}_u \times \underline{r}_v| \Delta u \Delta v$

scalar function defined on S

$$\rightsquigarrow \iint_S f \, dS := \lim_{\max(\Delta u_i, \Delta v_j) \to 0} \sum_{i,j} f(P_{ij}^*) \cdot \text{area}(S_{ij})$$

$$\approx \lim_{\max(\Delta u_i, \Delta v_j) \to 0} \sum_{i,j} f(P_{ij}^*) \cdot |\underline{r}_u \times \underline{r}_v| \Delta u_i \Delta v_j$$

$$\Rightarrow \boxed{\iint_S f\, dS = \iint_D f(\underline{r}(u,v))\, |\underline{r}_u \times \underline{r}_v|\, dA.}$$

In particular, $\text{area}(S) = \iint_S 1\, dS = \iint_D 1\, |\underline{r}_u \times \underline{r}_v|\, dA.$

Ex 1. $I = \iint_S x^2\, dS$, $\quad S = \{\underbrace{x^2 + y^2 + z^2 = 1}_{\rho = 1}\}$.

A. $\underline{r}(\theta, \phi) := \langle \sin\phi \cos\theta,\ \sin\phi \sin\theta,\ \cos\phi \rangle$, $\quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$.

$$\underline{r}_\theta \times \underline{r}_\phi = \langle -\sin\phi \sin\theta,\ \sin\phi\cos\theta,\ 0\rangle \times \langle \cos\phi\cos\theta,\ \cos\phi \sin\theta,\ -\sin\phi\rangle$$

$$= \begin{vmatrix} i & j & k \\ -\sin\phi \sin\theta & \sin\phi \cos\theta & 0 \\ \cos\phi \cos\theta & \cos\phi \sin\theta & -\sin\phi \end{vmatrix}$$

$$= \langle -\sin^2\phi \cos\theta,\ -\sin^2\phi \sin\theta,\ \underbrace{-\sin\phi\cos\phi \sin^2\theta - \sin\phi\cos\phi \cos^2\theta}_{-\sin\phi\cos\phi} \rangle$$

$$\Rightarrow |\underline{r}_\theta \times \underline{r}_\phi| = \sqrt{\underbrace{\sin^4\phi \cos^2\theta + \sin^4\phi \sin^2\theta}_{\sin^4\phi} + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^4\phi + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^2\phi\,(\sin^2\phi + \cos^2\phi)}$$

$$= \sqrt{\sin^2\phi} = |\sin\phi| = \sin\phi$$

$$I = \iint_S x^2\, dS = \int_0^{2\pi}\int_0^{\pi} \sin^2\phi \cos^2\theta \cdot \sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \underline{\sin^3\phi} \cos^2\theta \, d\phi\, d\theta$$

$\sin\phi \cdot (1-\cos^2\phi)$

$u = \cos\phi$, $du = -\sin\phi\, d\phi$,

$\sin\phi\, d\phi = -du$

$$= \int_0^{2\pi}\int_{-1}^{1} (1-u^2)\cos^2\theta \, du\, d\theta$$

$$= \int_0^{2\pi} \Big[\cos^2\theta \left(u - \tfrac{1}{3}u^3\right)\Big]_{u=-1}^{u=1} d\theta$$

$$= \int_0^{2\pi} \frac{4}{3}\cdot\cos^2\theta\, d\theta = \frac{4\pi}{3}.$$

<u>Special case: $S = \{z = g(x,y)\}$.</u>

Say $S = \{z = g(x,y) \mid (x,y) \in D\}$.

Can parametrize by $r(u,v) := \langle u, v, g(u,v)\rangle$, $(u,v) \in D$.

Then
$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & g_u \\ 0 & 1 & g_v \end{vmatrix} = \langle -g_u, -g_v, 1\rangle$$

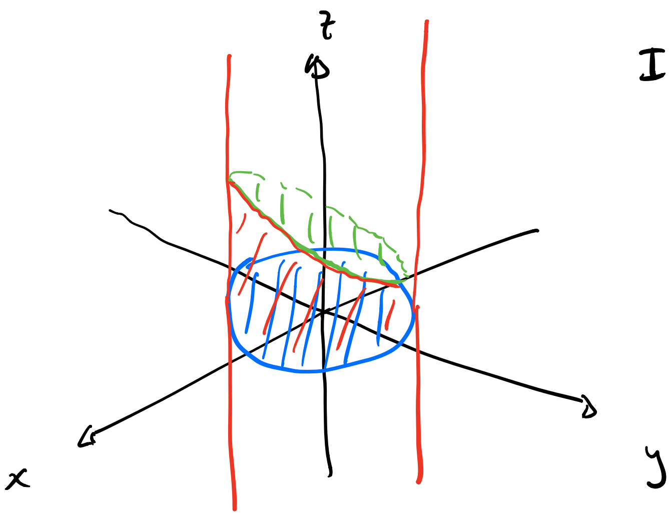
$$\Rightarrow |r_u \times r_v| = \sqrt{1 + g_u^2 + g_v^2}$$

$$\longrightarrow \boxed{\iint_S f\, dS = \iint_D f(u,v,g(u,v)) \cdot \sqrt{1+g_u^2+g_v^2}\, dA.}$$

Ex 3. $I = \iint_S z\, dS$, where S is surface whose sides are given by:

- S_1 : $\{x^2+y^2=1\}$
- S_2 : unit disk in xy-plane
- S_3 : part of the plane $\{z = \underbrace{1+x}_{g(x,y)}\}$.

a.



$$I = \iint_S z\, dS = \iint_{S_1} z\, dS + \iint_{S_2} z\, dS + \iint_{S_3} z\, dS.$$

$\iint_{S_3} z\, dS$? $\quad \underline{r}_3(u,v) := \langle u, v, 1+u \rangle$, $\{u^2+v^2 \le 1\}$

$$\rightsquigarrow \iint_{S_3} z\, dS = \iint_{\{u^2+v^2\le 1\}} (1+u)\cdot\sqrt{1+1^2+0^2}\, dA$$

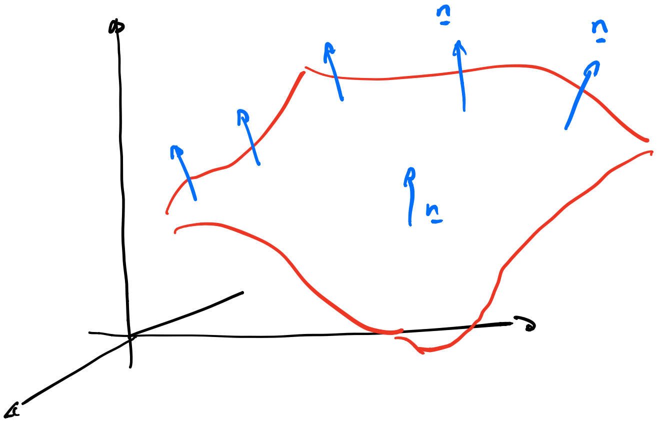
$$= \iint_{\{u^2+v^2\le 1\}} \sqrt{2}\cdot(1+u)\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}\cdot(1+r\cos\theta)\cdot r \, dr\, d\theta.$$

... Δ

Surface integrals of vector fields: $\iint_S \underline{F}\cdot d\underline{S}$.

think: rate of fluid flow through a surface.



note: can define a "unit normal v.f." to S by $\dfrac{\underline{r}_u \times \underline{r}_v}{|\underline{r}_u \times \underline{r}_v|} =: \underline{n}$.

$$\text{Def. } \iint_S \underline{F}\cdot d\underline{S} := \iint_S \underline{F}\cdot \underline{n}\, dS$$

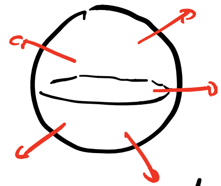
Note:

$$\iint_S \underline{F}\cdot\underline{n}\, dS = \iint_D F(\underline{r}(u,v))\cdot \frac{\underline{r}_u\times\underline{r}_v}{\cancel{|\underline{r}_u\times\underline{r}_v|}}\cdot \cancel{|\underline{r}_u\times\underline{r}_v|}\, dA$$

$$= \iint_D F(\underline{r}(u,v))\cdot(\underline{r}_u\times\underline{r}_v)\, dA$$

how we'll actually compute

§13.7 (cont)



Ex 4. Find flux of $\underline{F} = \langle z, y, x \rangle$ across the unit sphere $\{x^2+y^2+z^2=1\}$.

A. Recall:

$$= \langle \cos\phi,\ \sin\phi \sin\vartheta,\ \sin\phi\cos\vartheta \rangle$$

$$I = \iint_S \underline{F} \cdot d\underline{S} = \iint_D \underline{F}(\underline{r}(u,v)) \cdot (\underline{r}_u \times \underline{r}_v)\, dA.$$

$$\underline{r}(\theta, \phi) := \langle \sin\phi \cos\vartheta,\ \sin\phi \sin\vartheta,\ \cos\phi \rangle$$

$$\Rightarrow \underline{r}_\vartheta \times \underline{r}_\phi = \langle -\sin^2\phi \cos\vartheta,\ -\sin^2\phi \sin\vartheta,\ -\cos\phi \sin\phi \rangle$$

$$@\ \phi = \tfrac{\pi}{2},\ \vartheta = 0 \ : \ \underline{r}_\vartheta \times \underline{r}_\phi = \langle -1, 0, 0 \rangle$$

$$\Rightarrow I = -\int_0^{2\pi}\int_0^{\pi} \langle \cos\phi,\ \sin\phi \sin\vartheta,\ \sin\phi\cos\vartheta \rangle \cdot \langle -\sin^2\phi\cos\vartheta,\ -\sin^2\phi \sin\vartheta,\ -\cos\phi\sin\phi \rangle\, d\phi\, d\vartheta$$

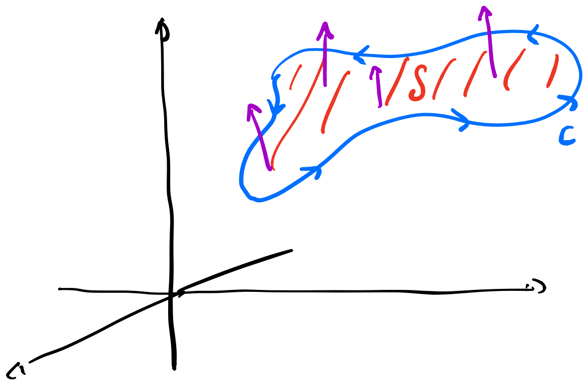
$$= -\int_0^{2\pi}\int_0^{\pi} \left(-\sin^2\phi\cos\phi\cos\vartheta - \sin^3\phi \sin^2\vartheta - \sin\phi\cos\phi\cos\vartheta\sin\vartheta \right) d\phi\, d\vartheta$$

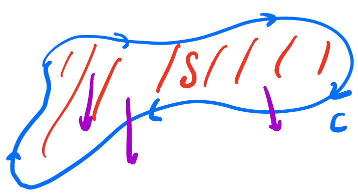
$\triangle$

§13.8: Stokes' theorem

Stokes' theorem: Let S be oriented, piecewise-smooth surface, bounded by a simple, closed, piecewise-smooth curve C, w/ positive orientation. Then:

$$\int_{C = \partial S} \underline{F} \cdot d\underline{r} = \iint_S \operatorname{curl} \underline{F} \cdot d\underline{S}.$$





~~Spring '18, # 8b.~~ $I = \int_C \underline{F} \cdot d\underline{r}$, $F = \langle -y^2, x, z^2 \rangle$,

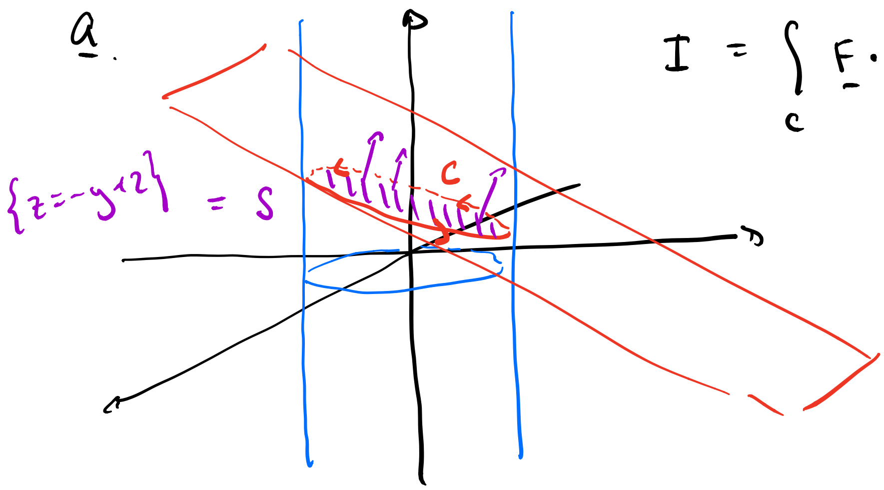
C = curve of intersection of $\{y + z = 2\}$, $\{x^2 + y^2 = 1\}$.

$z = -y + 2$

Orient C counterclockwise when viewed from above.

Compute I using Stokes' theorem.

a.



$$I = \int_C \underline{F} \cdot d\underline{r} = \iint_S \text{curl}\, \underline{F} \cdot d\underline{S}.$$

$$\text{curl}\, \underline{F} = \nabla \times \langle -y^2, x, z^2 \rangle = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \langle 0 - 0, -(0-0), 1 - (-2y) \rangle$$

$$= \langle 0, 0, 1 + 2y \rangle.$$

$\{z = -y + 2\} \rightsquigarrow \underline{r}(u,v) := \langle u, v, -v + 2 \rangle$, $\{u^2 + v^2 \le 1\}$.

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0, 1, 1 \rangle.$$

$$\rightsquigarrow \quad I = \int_C \underline{F} \cdot d\underline{r} \overset{S}{=} \iint_S \operatorname{curl} \underline{F} \cdot d\underline{S}$$

$$= \iint_D (\operatorname{curl} F)(\underline{r}(u,v)) \cdot (\underline{r}_u \times \underline{r}_v) \, dA$$

$$= \iint_D \langle 0, 0, 1+2v \rangle \cdot \langle 0, 1, 1 \rangle \, dA$$

$$= \iint_{\{u^2+v^2 \leq 1\}} (1+2v) \, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} r^2 + \frac{2}{3} r^3 \sin\theta \right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin\theta \right) d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{2}{3} \cos\theta \right]_0^{2\pi}$$

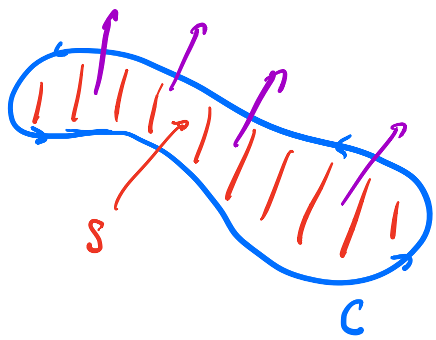
$$= \frac{1}{2} \cdot 2\pi = \pi. \qquad \triangle$$

Alternate final : 10pm PT, 11/18 (TBC).

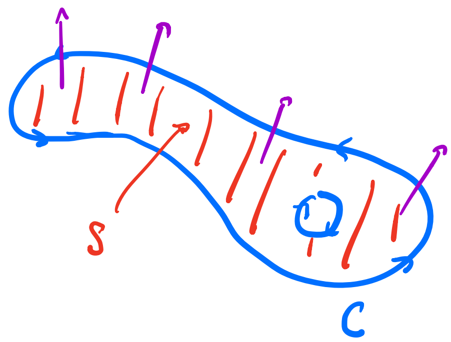
time for final : 120 mins.

Reviews : 11/15, 10-11 am, 11/16 6-7 pm.

<u>One thing from §13.8</u> :



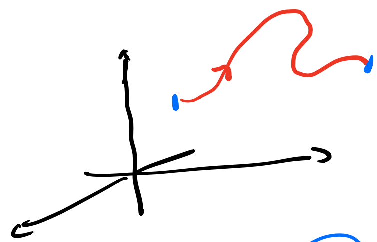
As we walk around C in the specified direction, w/ head pointing in direction specified by orientation on S, S should be on our left.



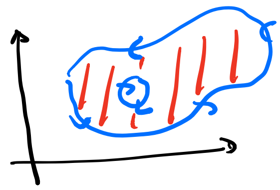
§13.9: The divergence theorem

FTC $\qquad \displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$

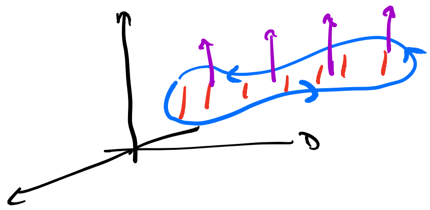
FTLI $\qquad \displaystyle\int_C \nabla f \cdot d\underline{r} = f(r(b)) - f(r(a))$



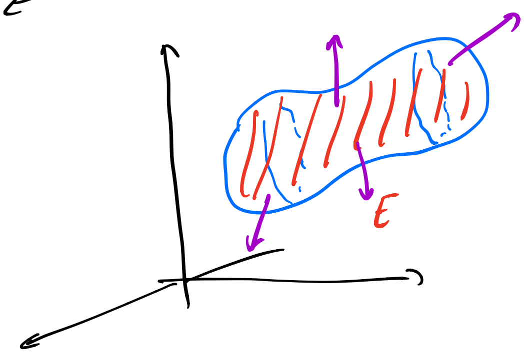
Green's $\qquad \displaystyle\iint_D (Q_y - P_x)\,dA = \int_{\partial D} (P\,dx + Q\,dy)$



Stokes' $\qquad \displaystyle\iint_S \operatorname{curl} \underline{F} \cdot d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{r}$



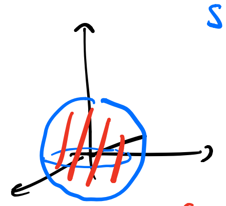
divergence $\qquad \displaystyle\iiint_E \operatorname{div} \underline{F}\,dV = \iint_{\partial E} \underline{F} \cdot d\underline{S}$



Divergence theorem. Let E be a solid region in $\mathbb{R}^3$ "with no holes".
Let S be boundary surface, w/ outward orientation.
Then:

$$\iiint_E \operatorname{div} \underline{F}\,dV = \iint_S \underline{F} \cdot d\underline{S}.$$

Ex 1. Find flux of $\underline{F} = \langle z, y, x \rangle$ through $\underbrace{\{x^2+y^2+z^2=1\}}_{S}$.



$E = \{x^2+y^2+z^2 \le 1\}$

a.

$$I = \iint_S \underline{F} \cdot d\underline{S}$$

$$\overset{D}{=} \iiint_E \underbrace{\operatorname{div} \underline{F}}_{*} \, dV$$

$* = \nabla \cdot \langle z, y, x \rangle$

$= \partial_x(z) + \partial_y(y) + \partial_z(x) = 1$

$$= \iiint_{\{x^2+y^2+z^2\}} 1 \, dV$$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^1 \int_0^{2\pi} \Big[-\rho^2 \cos\phi \Big]_{\phi=0}^{\phi=\pi} d\theta \, d\rho$$

$$= \int_0^1 \int_0^{2\pi} 2\rho^2 \, d\theta \, d\rho$$

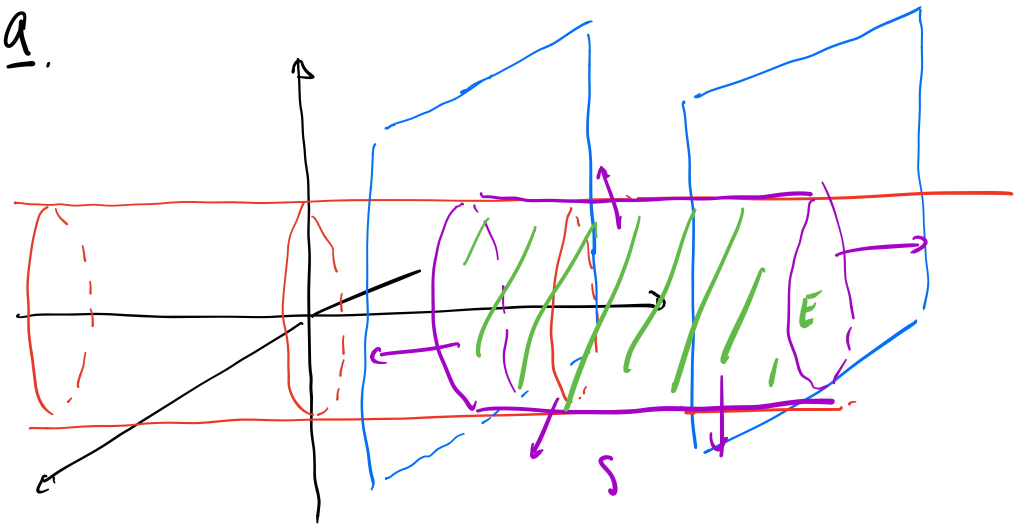
$$= \int_0^1 4\pi \rho^2 \, d\rho = \left[\frac{4\pi}{3} \rho^3 \right]_0^1 = \frac{4\pi}{3}.$$

Δ

??? #8. Evaluate $\iint_S \underline{F} \cdot d\underline{S}$, $\underline{F} = \langle y \sin z, 6x^2 y, 2z^3 \rangle$,

S boundary of solid bounded by $\{x^2+z^2=2\}$, $\{y=1\}$, $\{y=2\}$.
E

a.



$$I = \iint_S \underline{F} \cdot d\underline{S}$$

$$\overset{D}{=} \iiint_E \underline{\operatorname{div} \underline{F}} \, dV$$

$$= \partial_x (y \sin z) + \partial_y (6x^2 y) + \partial_z (2z^3)$$
$$= 0 + 6x^2 + 6z^2$$
$$= 6(x^2+z^2)$$

$$= \iiint_{\left\{\substack{x^2+z^2 \le 2 \\ 1 \le y \le 2}\right\}} 6(x^2+z^2) \, dV$$

use cylindrical coords w/ $x = r\cos\theta$, $y = y$, $z = r \sin\theta$

$$= \int_1^2 \int_0^{\sqrt{2}} \int_0^{2\pi} 6r^3 \, d\theta \, dr \, dy$$

$$= \int_1^2 \int_0^{\sqrt{2}} 12\pi r^3 \, dr \, dy$$

$$= \int_1^2 \left[3\pi r^4 \right]_{r=0}^{\sqrt{2}} dy = \int_1^2 12\pi \, dy$$

$$= 12\pi .$$

$\triangle$

- $f(x,y,z) \rightsquigarrow \nabla f$ a 3D vector field

- $\nabla f(x,y,z)$ is in direction of fastest increase of f; magnitude is that fastest rate

- $S = \{ f(x,y,z) = 0 \}$, p_0 on S $\rightsquigarrow$ $\nabla f(p_0)$ is perpendicular to S @ p_0.

Fall '13, #3. $f(x,y) = x^2 + xy + y^2$.

(a) Find tangent plane to $z = f(x,y)$ @ $(x,y) = (1,2)$

P

(c) In which direction is f increasing the fastest @ $(x,y) = (1,2)$?

A. (a) $S = \{ \underbrace{z - f(x,y)}_{g(x,y,z)} = 0 \}$.

Note, $z_0 = f(1,2)$
$= 1 + 2 + 4 = 7$

$\rightsquigarrow p_0 = (1,2,7)$.

Normal vector to P: $\nabla g(1,2,7)$.

$$\nabla g = \langle -f_x, -f_y, 1 \rangle = \langle -2x - y, -x - 2y, 1 \rangle$$

$$\Rightarrow \nabla g(1,2,7) = \langle -4, -5, 1 \rangle$$

$\Rightarrow$ P is given by $\langle -4, -5, 1 \rangle \cdot \left(\langle x,y,z \rangle - \langle 1,2,7 \rangle \right) = 0$

$$\Longleftrightarrow -4(x-1) - 5(y-2) + 1 \cdot (z-7) = 0.$$

(c) direction is $\dfrac{\nabla f(1,2)}{|\nabla f(1,2)|} = \dfrac{\langle 4,5 \rangle}{|\langle 4,5 \rangle|} = \dfrac{1}{\sqrt{41}} \langle 4,5 \rangle$.

Δ

tangent plane to $\{z = f(x,y)\}$ @ $(x_0, y_0, f(x_0,y_0))$:

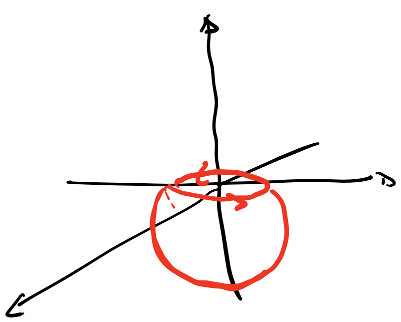
$$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

<u>Spring '14, 9b.</u> $S :=$ part of $\{x^2+y^2+(z+\sqrt{3})^2 = 4\}$ lying below xy-plane. (*)

(a) Show that bdry curve of S is unit circle $\{x^2+y^2=1\}$ in xy-plane.

(b) Evaluate $\iint_S \text{curl}\, \underline{F} \cdot d\underline{S}$, $\underline{F} = \langle y, -x, e^{xz} \rangle$.

<u>a</u>.



(a) sub $z=0$ into (*): $x^2+y^2+3=4$
$\Leftrightarrow x^2+y^2=1$.

(b)

✓ Stokes': $\iint_S \text{curl}\, \underline{F} \cdot d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{r}$

✗ divergence: $\iiint_E \text{div}\, \underline{F}\, dV = \iint_{\partial E} \underline{F} \cdot d\underline{S}$

$$I = \iint_S \text{curl}\, \underline{F} \cdot d\underline{S} = \int_{\{x^2+y^2=1\} \text{ in xy-plane}} \underline{F} \cdot d\underline{r}.$$

parametrize C by: $\underline{r}(t) := \langle \cos t, \sin t, 0 \rangle$, $0 \le t \le 2\pi$.

$$\underline{F} = \langle y, -x, e^{xz} \rangle$$

$$I = \int_C \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \langle \sin t, -\cos t, 1 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

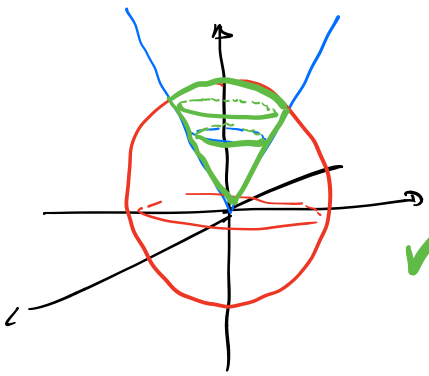
$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t) \, dt$$

$$= \int_0^{2\pi} (-1) \, dt = -2\pi. \quad \Delta$$

<u>Fall '17, #10.</u> $E :=$ points above $z = 2\sqrt{x^2+y^2}$, inside $x^2+y^2+z^2 = a^2$.

Compute $I = \iint_S \underline{F} \cdot d\underline{S}$, $\underline{F} = \langle x^3 z + 2xy, xz^2 - y^2, x^2 z^2 \rangle$,

S the bdry of E, oriented outward.

a. 

Stokes' : $\iint_S \text{curl } \underline{F} \cdot d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{r}$

✓ divergence : $\iiint_E \text{div } \underline{F} \, dV = \iint_{\partial E} \underline{F} \cdot d\underline{S}$

$$\text{div } \underline{F} = (3x^2 z + \cancel{2y}) + (-\cancel{2y}) + (2x^2 z)$$

$$= 5x^2 z.$$

D $\Rightarrow$ $I = \iiint_E 5x^2 z \, dV$

What is I in spherical coordinates?

- $x^2+y^2+z^2 = a^2 \iff \rho = a.$
- $z = 2\sqrt{x^2+y^2} \implies \rho\cos\phi = 2\sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

$$= 2\sqrt{\rho^2\sin^2\phi}$$

$$= 2\rho\sin\phi$$

$$\implies \tan\phi = \frac{1}{2}$$

$$\implies \phi = \pi/6.$$

$$\implies \boxed{E = \left\{ \begin{array}{l} 0 \le \rho \le a \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \tan^{-1}(1/2) \end{array} \right\}.}$$

Δ