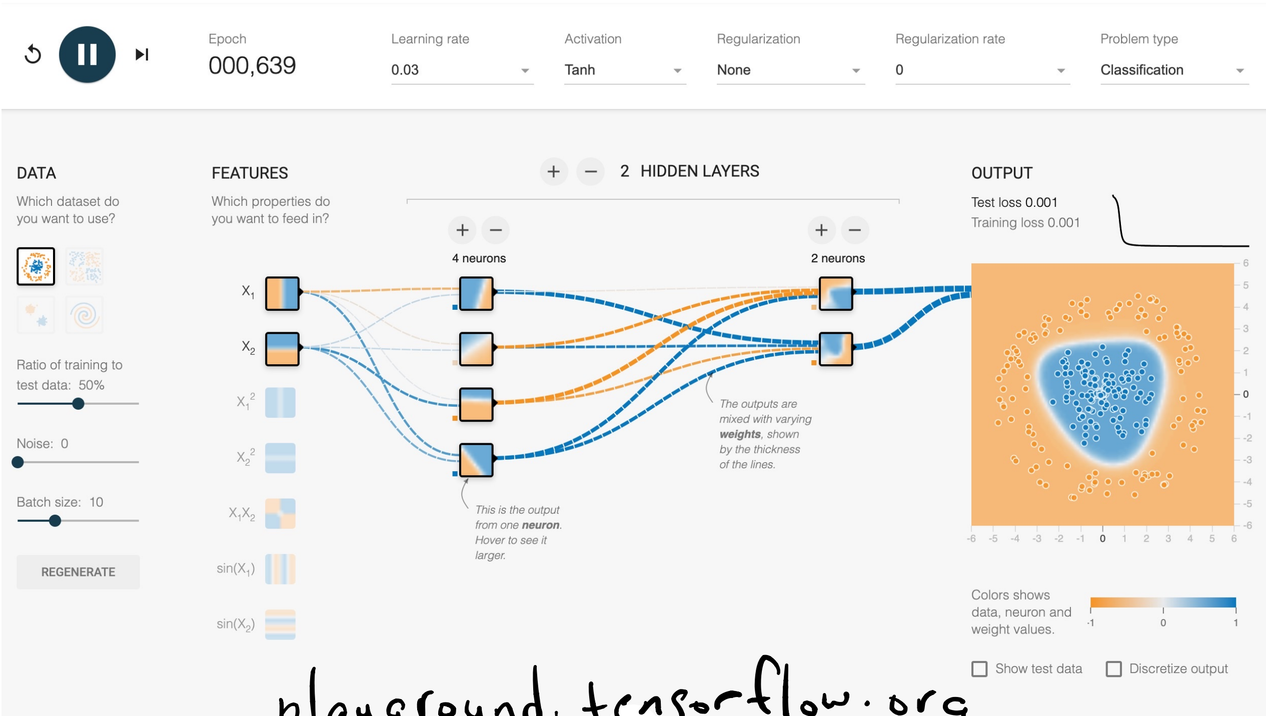
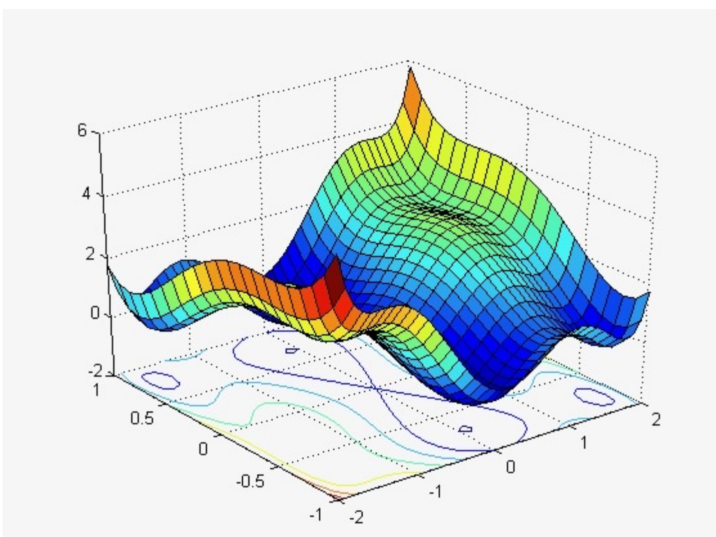
Lecture 1 of MATH 226

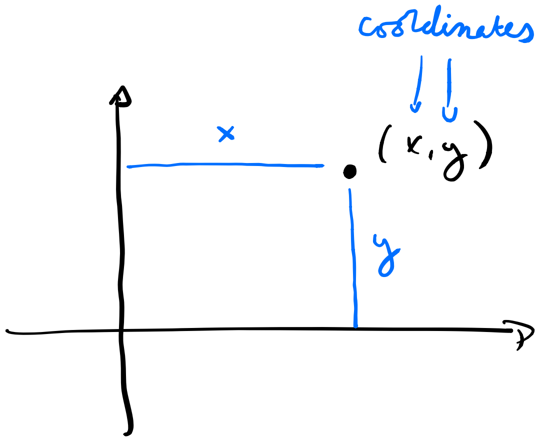


playground.tensorflow.org

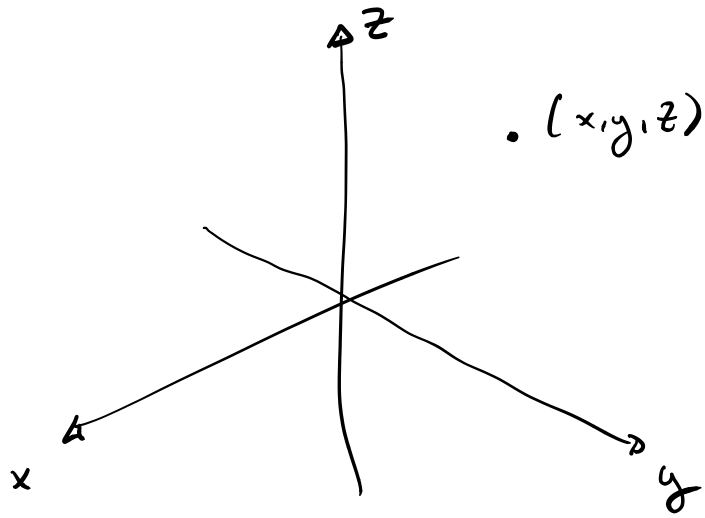


§10 : Vectors and the geometry of space.

§10.1 : 3-dimensional coordinate systems.



plane ($\mathbb{R}^2$)



3-dimensional space
(3-space, $\mathbb{R}^3$)

<u>Right-hand rule</u>: to determine which way the +z-axis is, curl fingers from +x-axis to +y-axis. Thumb points in direction of +z-axis.

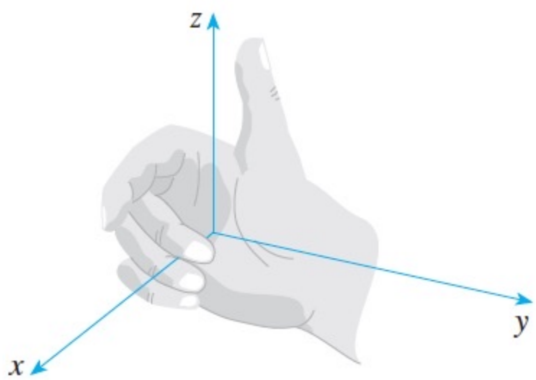
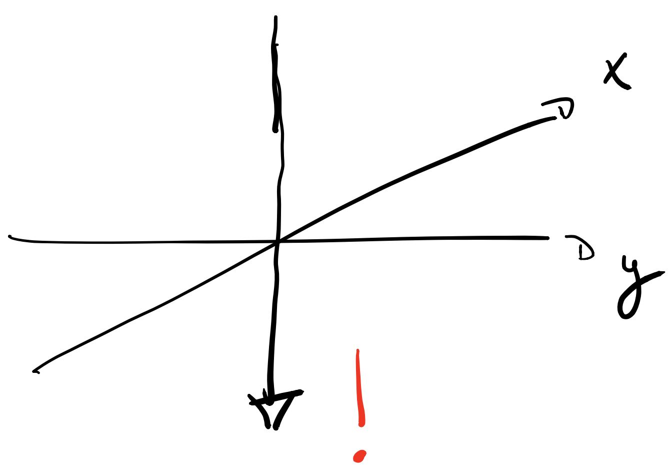


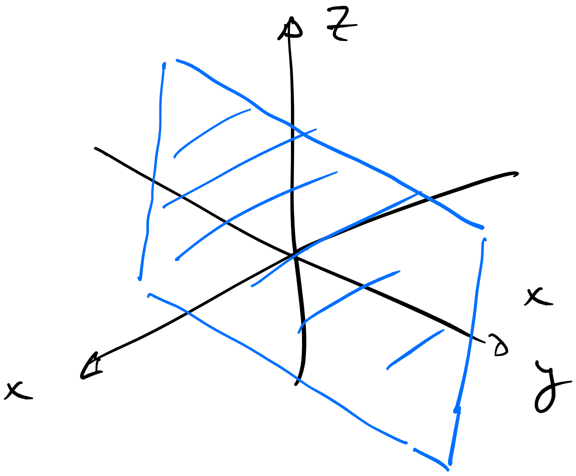
FIGURE 2
Right-hand rule

<u>Ex</u>.

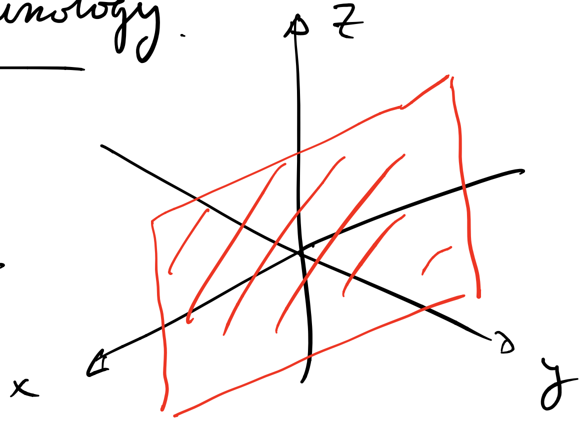


More basic terminology.

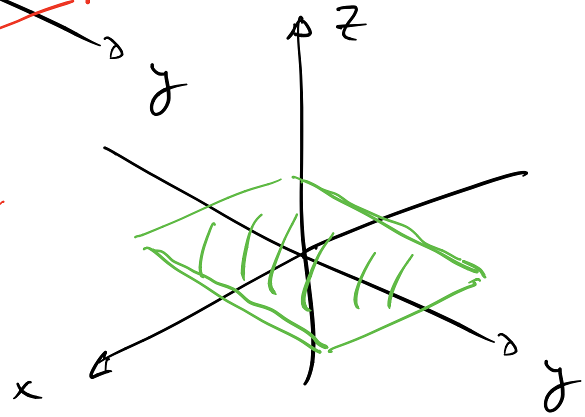
coordinate planes.



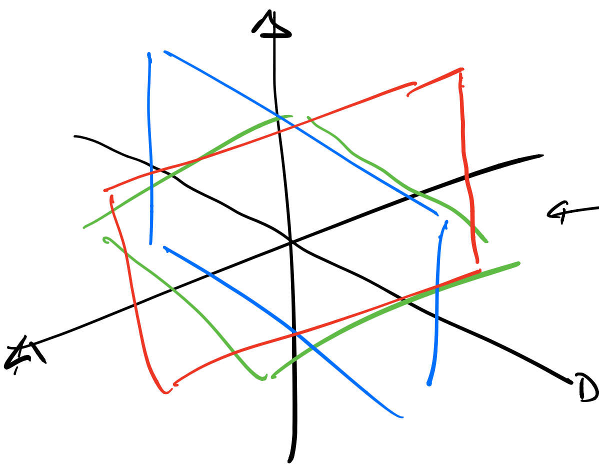
yz-plane



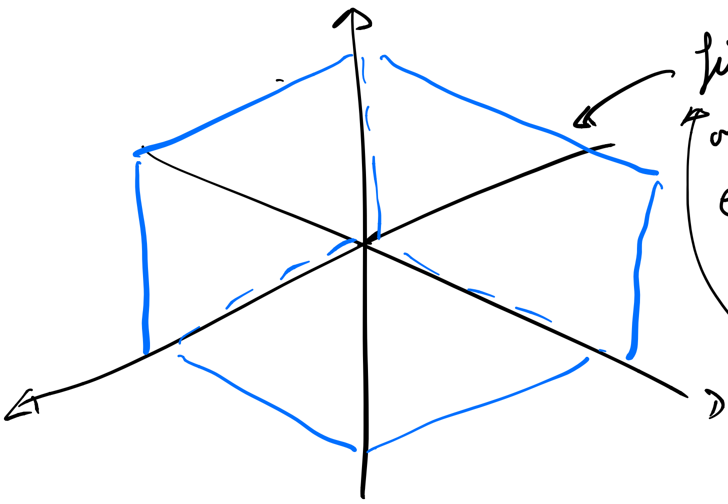
xz-plane.



xy-plane



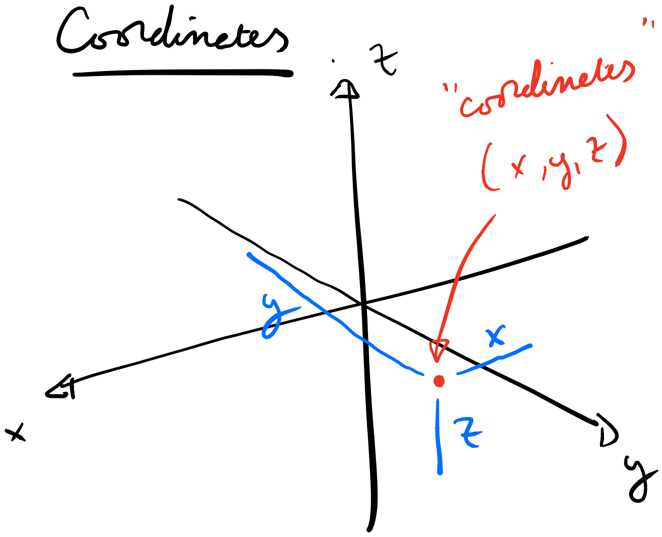
coordinate planes divide $\mathbb{R}^3$ into 8 <u>octants</u>.

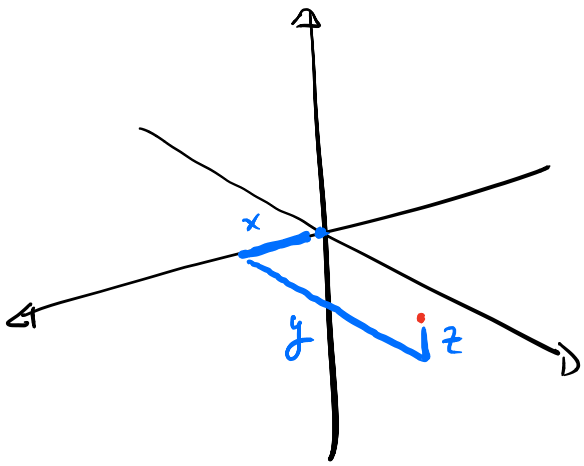


first octant: region on "positive side" of each coordinate plane.

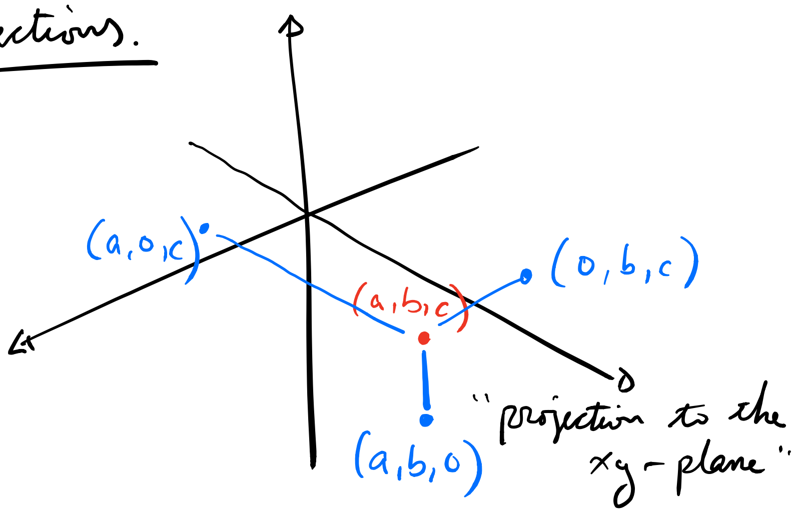
$$\left\{ (x,y,z) \;\middle|\; \begin{array}{l} x \geq 0, \\ y \geq 0, \\ z \geq 0 \end{array} \right\}.$$

Coordinates





Projections.



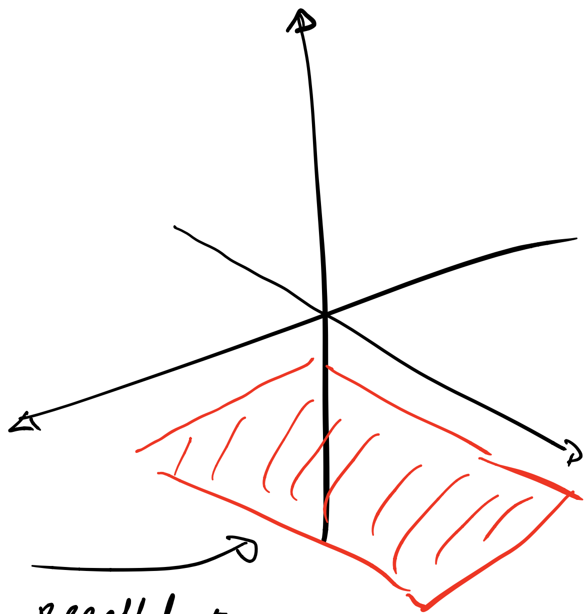
Q. Draw the following sets:

(a) $\{(x,y,z) \mid z=-3\}$

(b) $\{(x,y,z) \mid x=y\}$ ☆

(a) $z=-3$

$\Longleftrightarrow$ (x,y,z) is 3 units <u>below</u> xy-plane.



plane parallel to xy-plane, 3 units <u>below</u> it.

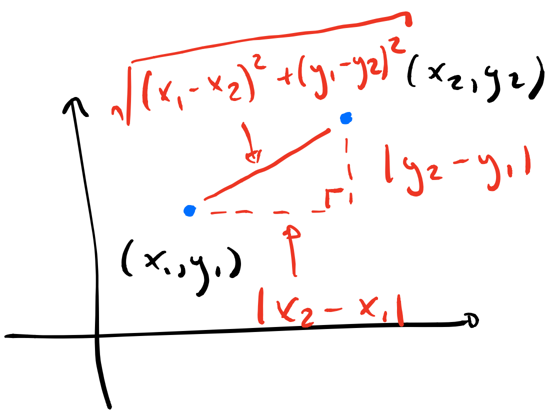
<u>Distance formula in $\mathbb{R}^3$</u>.

$P_1 = (x_1, y_1, z_1)$

$P_2 = (x_2, y_2, z_2)$

$$\boxed{|P_1P_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}}$$

(in $\mathbb{R}^2$:



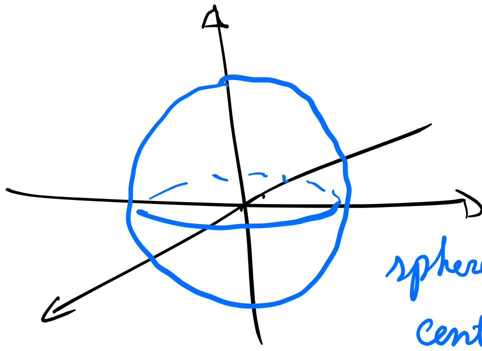
)

Ex. What is the surface defined by $x^2+y^2+z^2=9$?

A. $x^2+y^2+z^2 = (x-0)^2+(y-0)^2+(z-0)^2$

$=$ distance from (x,y,z) to $(0,0,0)$

$\rightarrow \{x^2+y^2+z^2=9\} = \{(x,y,z) \mid \text{distance of 3 from the origin}\}$



sphere of radius 3, centered @ $(0,0,0)$.

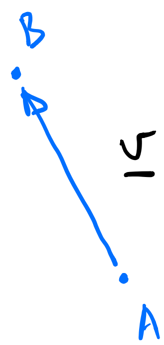
§10.2: Vectors

A vector is something with magnitude, direction.

wind speed — wind direction

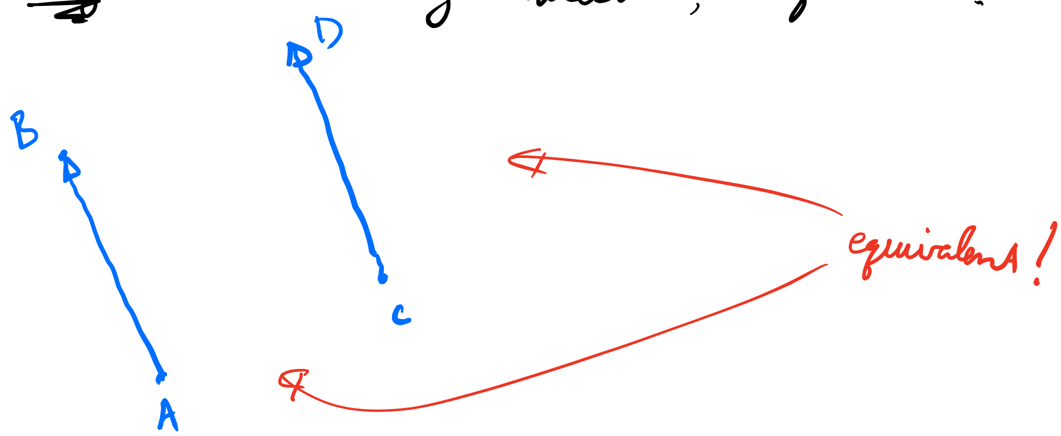
magnitude of force — direction of force

Fundamental example of vector: displacement vector as an object moves from A to B.



$\underline{v} = \overrightarrow{AB}$

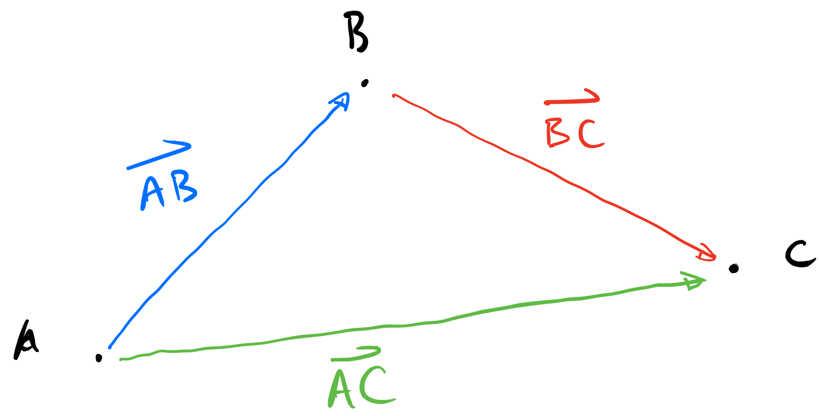
A vector is only the data of direction, magnitude.



$\rightsquigarrow$ if two vectors differ by a shift, they are equivalent!

Manipulating vectors

Say particle moves from A to B, then B to C.



$$\overrightarrow{AC} := \overrightarrow{AB} + \overrightarrow{BC}.$$

$\boxed{\text{see below}}$

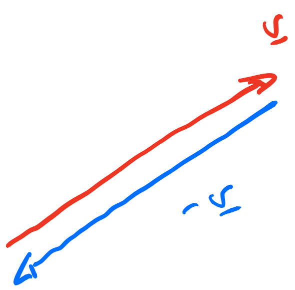
general definition of vector summation:

to form $\underline{v} + \underline{w}$, put tail of $\underline{w}$ at the head of $\underline{v}$, $\underline{v} + \underline{w}$ goes from tail of $\underline{v}$ to the head of $\underline{w}$.

Q: What is "$-\underline{v}$"?

A. Well, we should have $\underline{v} + (-\underline{v}) = \underline{0}$.

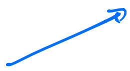
"0-vector", ie length-0 vector.

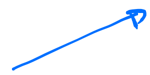


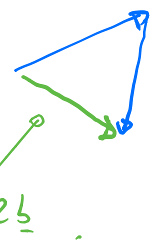
Def. $-\underline{v}$ has same length as $\underline{v}$, points in opposite direction.

Def. Say c is a scalar. real number

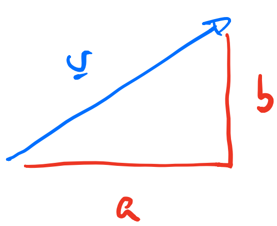
$$c\underline{v} := \begin{cases} \text{vector in the same direction as } \underline{v}, \text{ length is } c\cdot\left(\text{length of } \underline{v}\right), & c > 0 \\ \text{vector in opposite direction as } \underline{v}, \text{ length is } -c\cdot\left(\text{length of } \underline{v}\right), & c < 0 \end{cases}$$

Ex. $\underline{a} =$  $\underline{b} =$ . What's $\underline{a} - 2\underline{b}$?

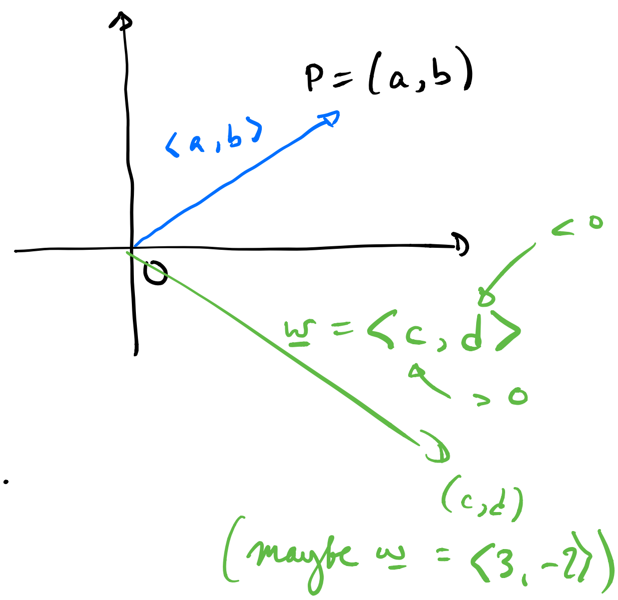
A. $\underline{a} - 2\underline{b} = \underline{a} + (-2\underline{b}) =$  $+$ 

$=$ 

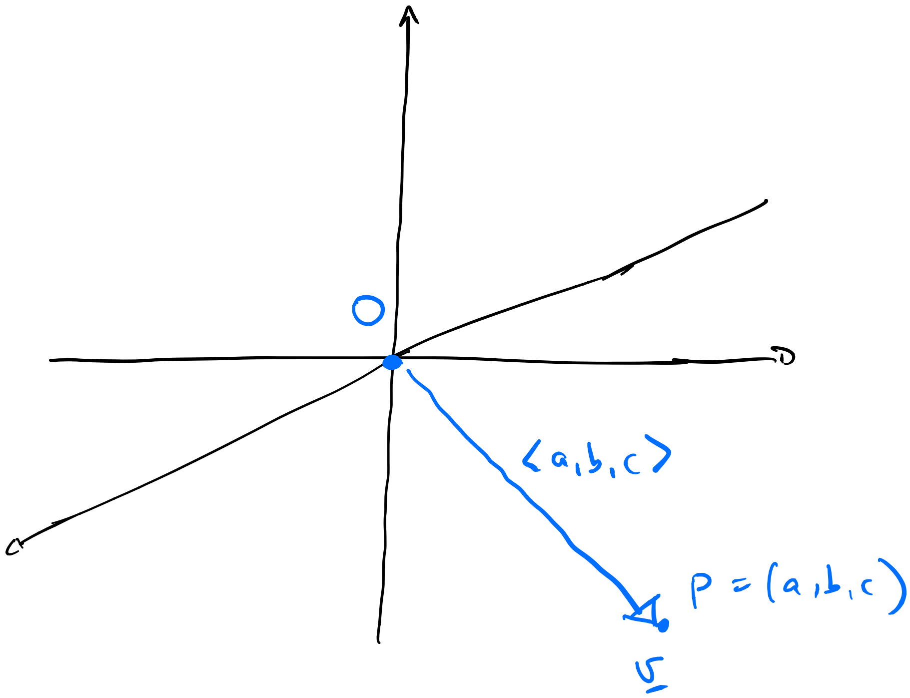
Components.

in 2D. 

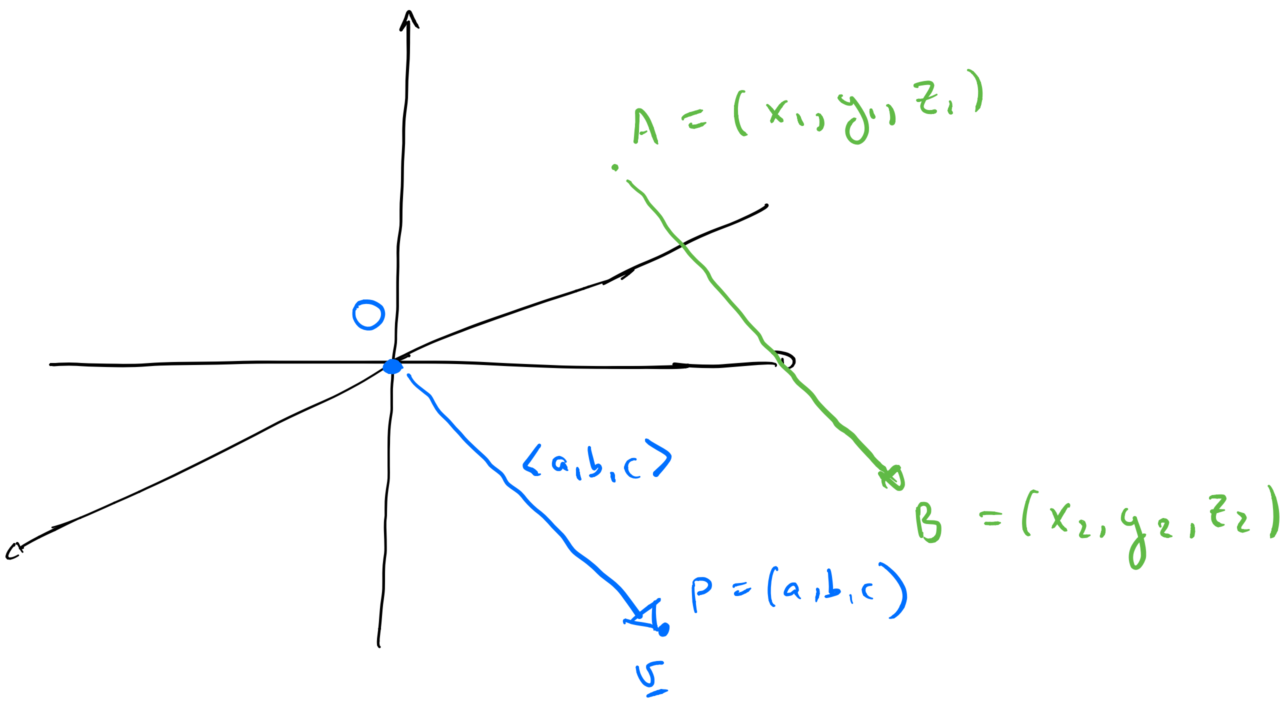
$\rightsquigarrow \underline{v} = \langle a, b \rangle$.



in 3D.



Q. In terms of components, when are 2 vectors equivalent?



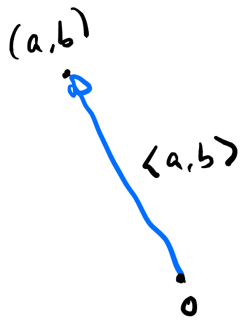
$$\vec{OP} = \vec{AB} \iff x_2 = x_1 + a,\ y_2 = y_1 + b,\ z_2 = z_1 + c$$

$$\iff x_2 - x_1 = a,\ y_2 - y_1 = b,\ z_2 - z_1 = c$$

In particular, in components, the vector from $A = (x_1, y_1, z_1)$ to $B = (x_2, y_2, z_2)$ is $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

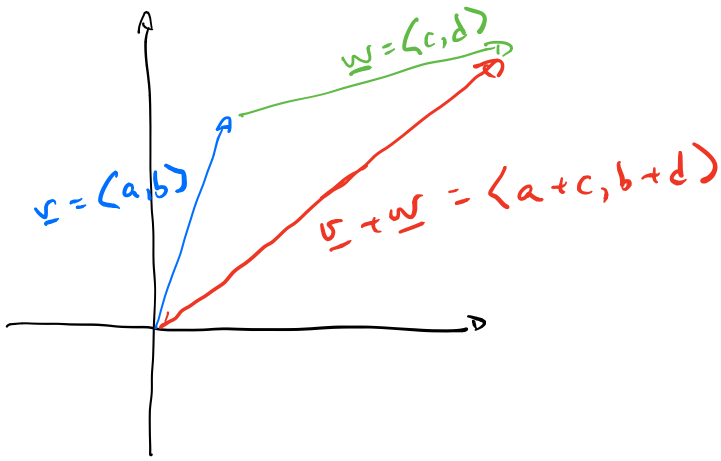
The **magnitude** $|\underline{v}|$ of $\underline{v} = \langle a, b, c \rangle$ is $|\underline{v}| = \sqrt{a^2 + b^2 + c^2}$

$$|\langle a, b \rangle| = \sqrt{a^2 + b^2}.$$



length = distance between 0, (a,b)
$= \sqrt{a^2 + b^2}$.

Manipulation of vectors, in terms of components.



Say $\underline{v} = \langle a, b \rangle$, $\underline{w} = \langle c, d \rangle$.

- $\underline{v} + \underline{w} = \langle a + c, b + d \rangle$
- $\underline{v} - \underline{w} = \langle a - c, b - d \rangle$
- $\alpha \underline{v} = \langle \alpha a, \alpha b \rangle$.

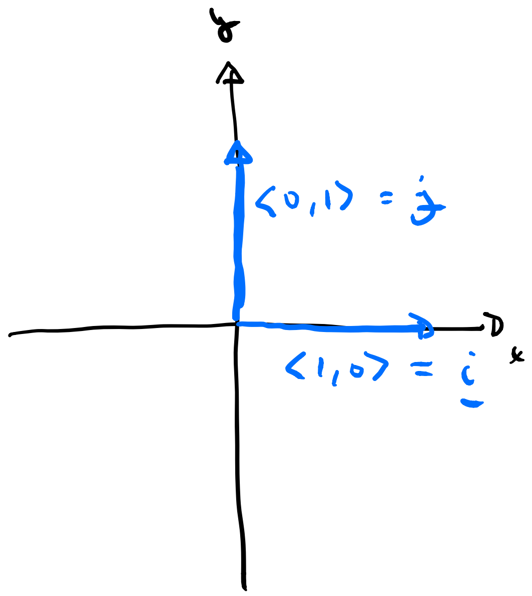
scalar.

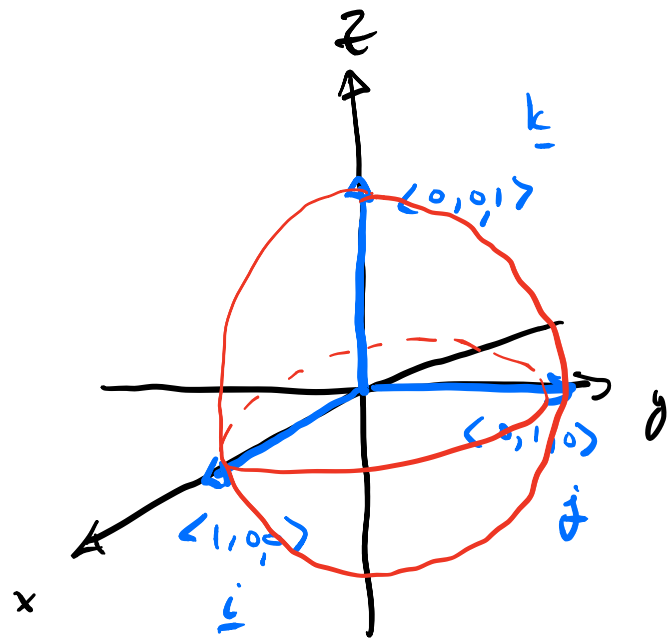
a <u>unit vector</u> is a $\underline{v}$ w/ $|\underline{v}| = 1$.

<u>Q</u>. Given $\underline{v}$, how to find unit vector w/ same direction?

OH today: 2-3; HW 1 posted.

Unit basis vectors.





$$\langle a,b,c\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$$
$$= a\,\underline{i} + b\,\underline{j} + c\,\underline{k}.$$

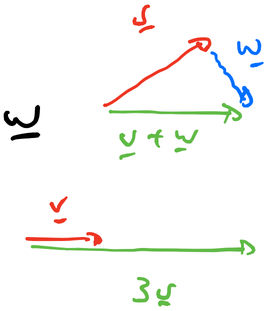
Q: Given $\underline{v}$, what's a <u>unit</u> vector in same direction?

A: $\boxed{\frac{1}{|\underline{v}|}\,\underline{v}}$.

§10.3: $\cdot$

We've seen some operations on vectors...

* $\underline{v} + \underline{w}$
* $c\,\underline{v}$



Multiplication of vectors??? Maybe $\langle a,b\rangle\langle c,d\rangle := \langle ac, bd\rangle$

No!

We'll see 2 answers... 1st: dot product!

Def. The dot product is defined by...

$$\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle := a_1b_1 + a_2b_2$$

$$\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle := a_1b_1 + a_2b_2 + a_3b_3$$

scalar

2 vectors of same dimension

($A := B$ means "A is defined to be B")

Ex.

$$\langle 4, -5\rangle \cdot \langle 0, 3\rangle = 4\cdot 0 + (-5)\cdot 3 = -15.$$

$\underline{i} + \underline{j} + 0\underline{k}$

$$(\underline{i} + \underline{j}) \cdot (-2\underline{j} + \underline{k}) = 1\cdot 0 + 1\cdot(-2) + 0\cdot 1 = -2.$$

$\langle 1,1,0\rangle$ $\quad\langle 0,-2,1\rangle$

$0\,\underline{i} - 2\underline{j} + \underline{k} = \langle 0, -2, 1\rangle$

HANDY PROPERTIES OF $\cdot$

1. $\underline{a} \cdot \underline{a} = |\underline{a}|^2$.

$\left(\langle a_1, a_2 \rangle \cdot \langle a_1, a_2 \rangle = a_1^2 + a_2^2 = |\underline{a}|^2. \right)$

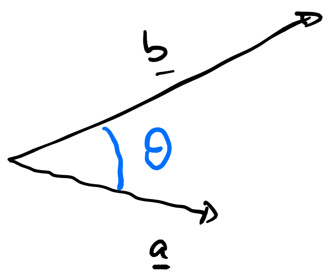
2. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

3. $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$.

4. $(\alpha \underline{a}) \cdot \underline{b} = \alpha (\underline{a} \cdot \underline{b}) = \underline{a} \cdot (\alpha \underline{b})$.

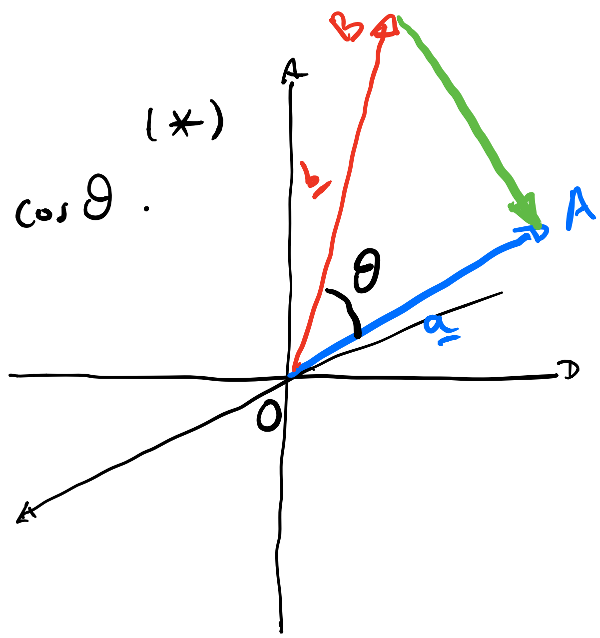
5. $\underline{0} \cdot \underline{a} = 0$.

WHY WE CARE ABOUT $\cdot$!



Let θ be the angle between $\underline{a}$, $\underline{b}$. Then:

$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta.} \quad (*)$$



Proof: Use law of cosines. $(*)$

$$\overset{|\underline{a}-\underline{b}|^2}{|BA|^2} = \overset{|\underline{a}|^2}{|OA|^2} + \overset{|\underline{b}|^2}{|OB|^2} - 2\,\overset{|\underline{a}|}{|OA|}\,\overset{|\underline{b}|}{|OB|} \cos\theta.$$

Note, $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$.

Also, $\overrightarrow{OB} + \overrightarrow{BA} = \overrightarrow{OA}$

$$\Rightarrow \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \underline{a} - \underline{b}.$$

$(*) \Rightarrow \underline{|a-b|^2} = |a|^2 + |b|^2 - 2|a||b|\cos\theta$

$(a-b)\cdot(a-b)$

$= a\cdot a - a\cdot b - b\cdot a + b\cdot b$

$= |a|^2 + |b|^2 - 2\,a\cdot b$

$\Rightarrow \cancel{|a|^2} + \cancel{|b|^2} - 2\,a\cdot b = \cancel{|a|^2} + \cancel{|b|^2} - 2|a||b|\cos\theta$

$\Rightarrow a\cdot b = |a||b|\cos\theta.$ W

Ex. Angle between $a = \langle 2,2,-1\rangle$, $b = \langle 5,-3,2\rangle$?

A. $a\cdot b = |a||b|\cos\theta$

$\Rightarrow \cos\theta = \dfrac{a\cdot b}{|a||b|}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{a\cdot b}{|a||b|}\right)$

$a\cdot b$: $2\cdot 5 + 2\cdot(-3) + (-1)\cdot 2 = 2.$

$|a||b|$: $= \sqrt{4+4+1}\cdot\sqrt{25+9+4}$
$= 3\sqrt{38}.$

$= \boxed{\cos^{-1}\left(\dfrac{2}{3\sqrt{38}}\right) \approx 84^\circ.}$

Some special cases.

$$-1 \le \cos\theta \le 1$$

$$\Rightarrow -|\underline{a}||\underline{b}| \le \underline{a}\cdot\underline{b} \le |\underline{a}||\underline{b}|.$$

What does it mean when $\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|, -|\underline{a}||\underline{b}|, 0$?

(1) $\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}| \iff \cos\theta \; 1$

$\iff \theta = 0$

$\iff \underline{a}, \underline{b}$ in same direction.

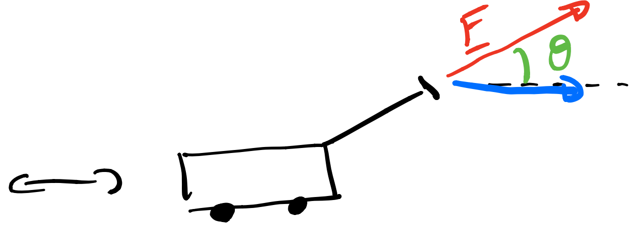
(2) $\underline{a}\cdot\underline{b} = -|\underline{a}||\underline{b}| \iff \theta = \pi$

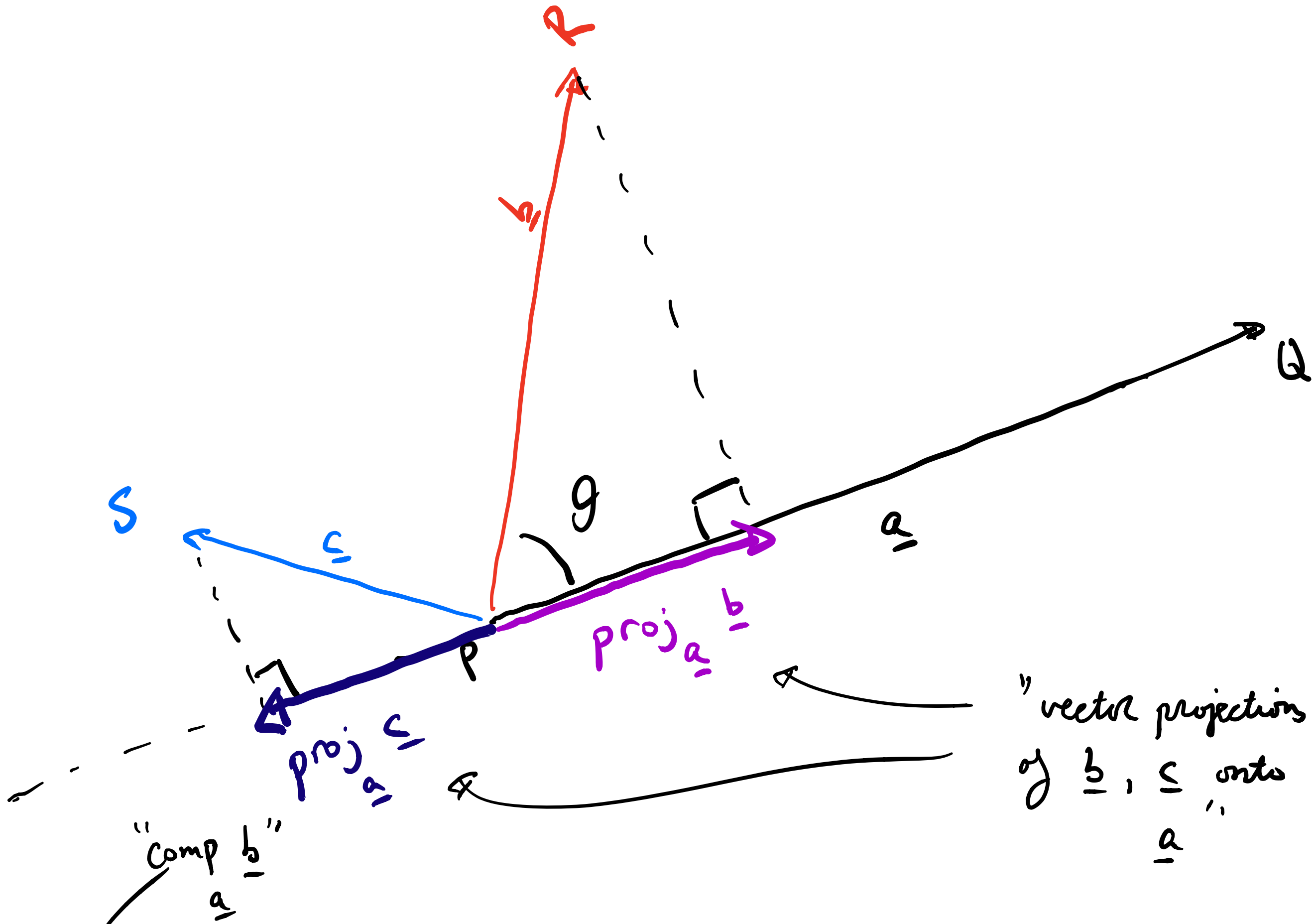
$\iff \underline{a}, \underline{b}$ in opp. direction.

(3) $\underline{a}\cdot\underline{b} = 0 \iff \theta = \pi/2$

$\iff \underline{a}, \underline{b}$ perpendicular / ortho-gonal

Projections.





Scalar projection is signed length of vector projection.

$$\text{comp}_{\underline{a}}\, \underline{b} = |\underline{b}| \cos\theta$$

$$= \frac{|\underline{a}||\underline{b}| \cos\theta}{|\underline{a}|}$$

$$= \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|}.$$

similarly, $\text{proj}_{\underline{a}}\,\underline{b} = \left(\text{comp}_{\underline{a}}\,\underline{b}\right)\left(\dfrac{\underline{a}}{|\underline{a}|}\right)$

$$= \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|}\cdot\frac{\underline{a}}{|\underline{a}|}$$

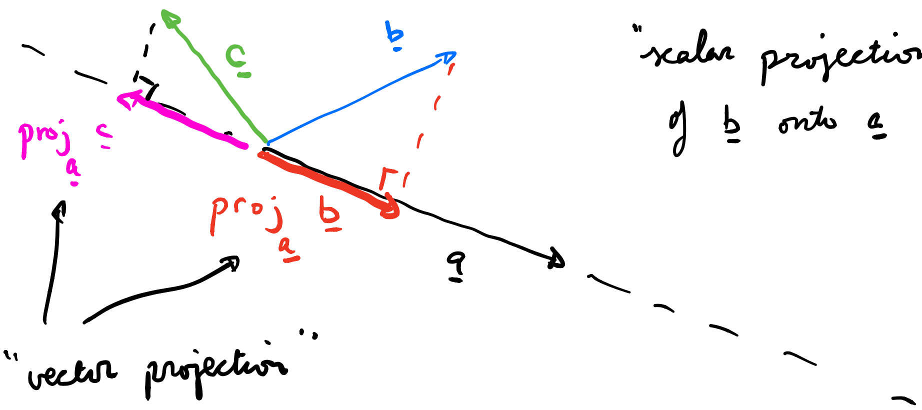
$$= \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\,\underline{a}\,.$$

end of §10.3:

Projections

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} := \begin{cases} |\text{proj}_{\mathbf{a}}\,\mathbf{b}|, & \text{proj}_{\mathbf{a}}\,\mathbf{b} \text{ in same direction as } \mathbf{a}, \\ -|\text{proj}_{\mathbf{a}}\,\mathbf{b}|, & \text{proj}_{\mathbf{a}}\,\mathbf{b} \text{ in opp. direction as } \mathbf{a}. \end{cases}$$

"scalar projection" of $\mathbf{b}$ onto $\mathbf{a}$

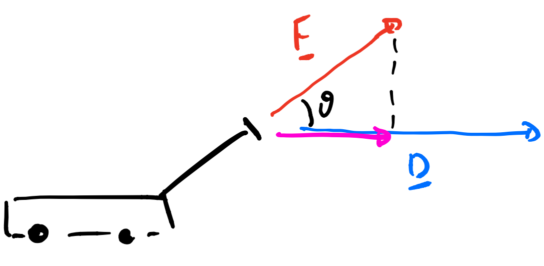


"vector projection"

$$\boxed{\begin{aligned} \text{comp}_{\mathbf{a}}\,\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}, \\ \text{proj}_{\mathbf{a}}\,\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a} \end{aligned}}$$

Work.

"work = force · distance" ← only works when force, displacement are in same direction!



$$\begin{aligned} W &:= (|\mathbf{F}|\cos\theta)\,|\mathbf{D}| \\ &= |\mathbf{D}|\,|\mathbf{F}|\cos\theta \\ &= \mathbf{F}\cdot\mathbf{D} \end{aligned}$$

§10.4: The cross product.

← 2nd answer to: what should we mean by "product of 2 vectors"?

Def. The cross product of two 3-dimensional vectors is:

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle := \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle.$$

(Sneak preview of why $\times$ is cool: $\underline{a} \times \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$.)

A way to remember $\times$: determinant. ← a number you can associate to a square grid of numbers.

$\boxed{2\times 2}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc.$$

$$\left(\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2. \right)$$

$\boxed{3\times 3}$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} := a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\left(\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \right.$$

$$= 1\cdot(45-48) - 2(36-42) + 3(32-35)$$

$$= -3 + 12 - 9$$

$$\left. \boxed{= 0.} \right)$$

Can remember $\times$ by:

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle$$

$$= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle$$

$$= \underline{i}\underbrace{(a_2b_3 - a_3b_2)}_{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}} - \underline{j}(a_1b_3 - a_3b_1) + \underline{k}(a_1b_2 - a_2b_1)$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

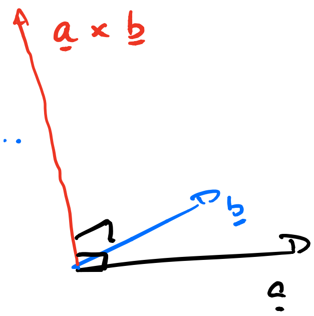
Ex 1. $\langle 1, 3, 4\rangle \times \langle 2, 7, -5\rangle = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$

$$= \underline{i}\underbrace{\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}}_{-15-28=-43} - \underline{j}\underbrace{\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}}_{-5-8=-13} + \underline{k}\underbrace{\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}}_{7-6=1}$$

$$= \langle -43, 13, 1\rangle.$$

Thm. $\underline{a} \times \underline{b}$ is orthogonal to both $\underline{a}$ and $\underline{b}$.

Pf. $(\underline{a} \times \underline{b} \perp \underline{a}) \iff (\underline{a} \times \underline{b}) \cdot \underline{a} = 0.$



$$(\underline{a} \times \underline{b}) \cdot \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot \underline{a}$$

$$= \underline{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \cdots$$

$$= a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= a_1(\cancel{a_2 b_3} - \cancel{a_3 b_2}) - a_2(\cancel{a_1 b_3} - \cancel{a_3 b_1}) + a_3(\cancel{a_1 b_2} - \cancel{a_2 b_1})$$

$$= 0.$$

In fact, $\underline{a} \times \underline{b}$ follows RHR: curl fingers of right hand from $\underline{a}$ to $\underline{b}$; thumb is in direction of $\underline{a} \times \underline{b}$.

Magnitude of $\underline{a} \times \underline{b}$?

$$|\underline{a} \times \underline{b}|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$= |\underline{a}|^2 |\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2 \qquad 1 - \cos^2\theta = \sin^2\theta$$

$$= |\underline{a}|^2 |\underline{b}|^2 - |\underline{a}|^2 |\underline{b}|^2 \cos^2\theta$$

$$= |\underline{a}|^2 |\underline{b}|^2 \sin^2\theta$$

$$\implies \boxed{|\underline{a} \times \underline{b}| = |\underline{a}| |\underline{b}| \sin\theta.}$$

(note: $0 \le \theta \le \pi$, so $\sin\theta \ge 0$.)

in same or opp. directions

Cor.

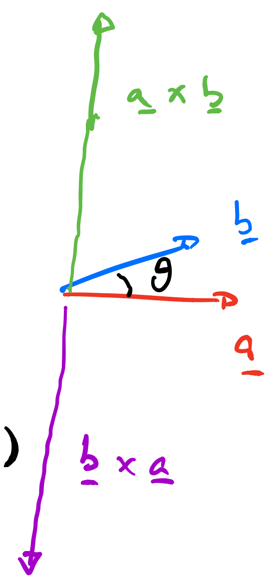
$\Longrightarrow$ $\underline{a}, \underline{b}$ are parallel if and only if $\underline{a} \times \underline{b} = \underline{0}$.

OH today: 3 - 4:30.

§10.4: $\times$ (con't).

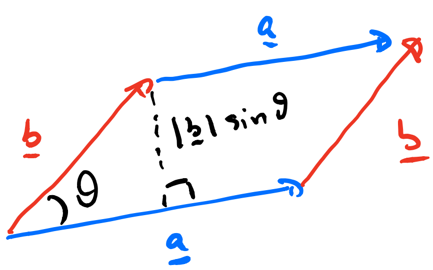
Recall: $\underbrace{\underline{a}, \underline{b}}_{\text{3-dim'l}} \rightsquigarrow \underbrace{\underline{a} \times \underline{b}}_{\text{3-dim'l}}$.

- $\underline{a} \times \underline{b}$ is orthogonal to $\underline{a}, \underline{b}$
- $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}| \sin\theta$ (*)



Can interpret (*) geometrically:

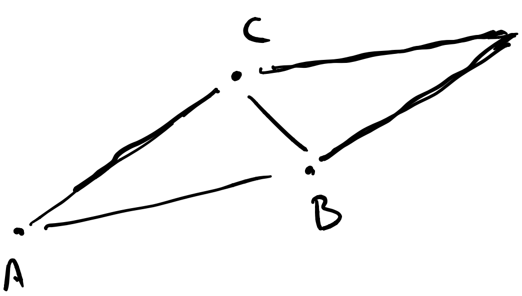
$\mathbb{R}^3$



$$\begin{aligned} \text{Area}(P) &= |\underline{a}| \cdot \left(|\underline{b}| \sin\theta\right) \\ &= |\underline{a}||\underline{b}| \sin\theta \\ &= |\underline{a} \times \underline{b}|. \end{aligned}$$

$\Longrightarrow$ $\boxed{|\underline{a} \times \underline{b}| = \text{area of parallelogram w/ sides } \underline{a}, \underline{b}.}$

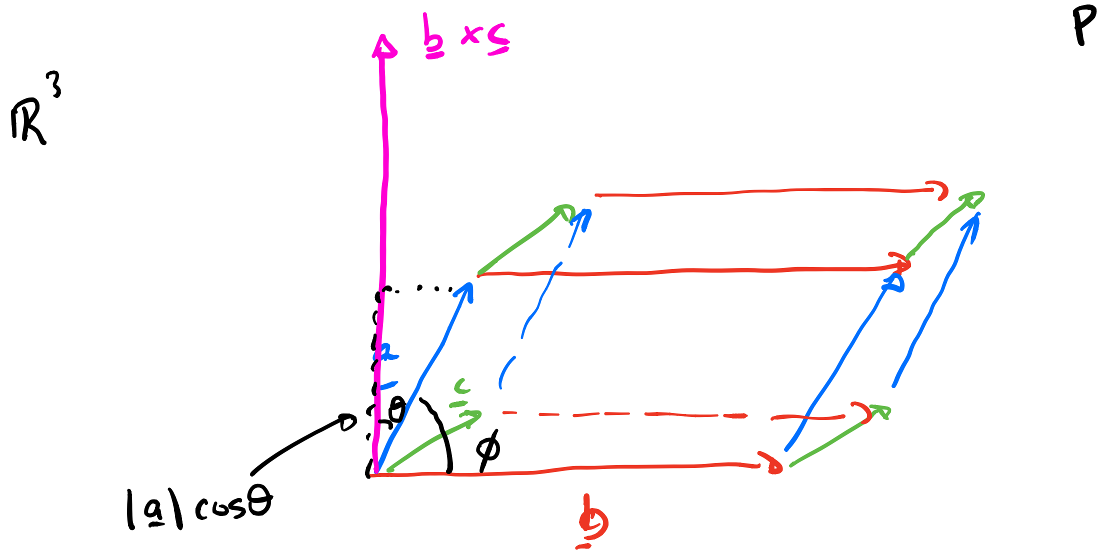
area of ΔABC
$= \frac{1}{2} |\vec{AB} \times \vec{AC}|$



$\mathbb{R}^3$.

Note: $\boxed{\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}.}$

The volume of a parallelepiped.



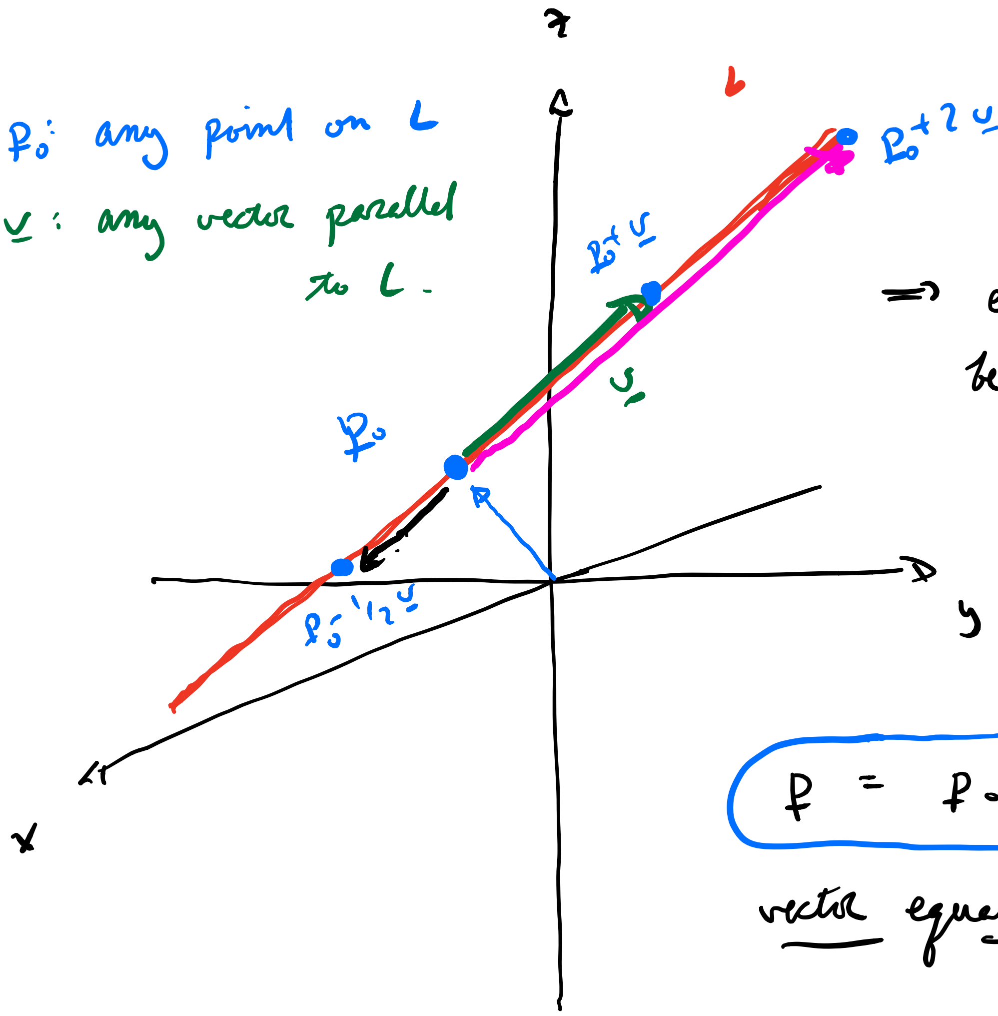
$$\text{Volume}(P) = \text{area(base)} \cdot \text{height}$$
$$= |\underline{b} \times \underline{c}| \cdot |\underline{a}| \cos\theta$$
$$= |\underline{a} \cdot (\underline{b} \times \underline{c})|.$$

torque X

§10.5: Equations of lines, planes in $\mathbb{R}^3$.

$\underline{p}_0$: any point on L

$\underline{v}$: any vector parallel to L.



$\Rightarrow$ every point on L can be described as

$$\underline{p}_0 + t\underline{v},$$

where t is some scalar.

$$(*) \quad \boxed{\underline{p} = \underline{p}_0 + t\underline{v}}$$

is the <u>vector equation of L</u>.

Write $\underline{p} = \langle x, y, z \rangle$, $\underline{p}_0 = \langle x_0, y_0, z_0 \rangle$, $\underline{v} = \langle a, b, c \rangle$.

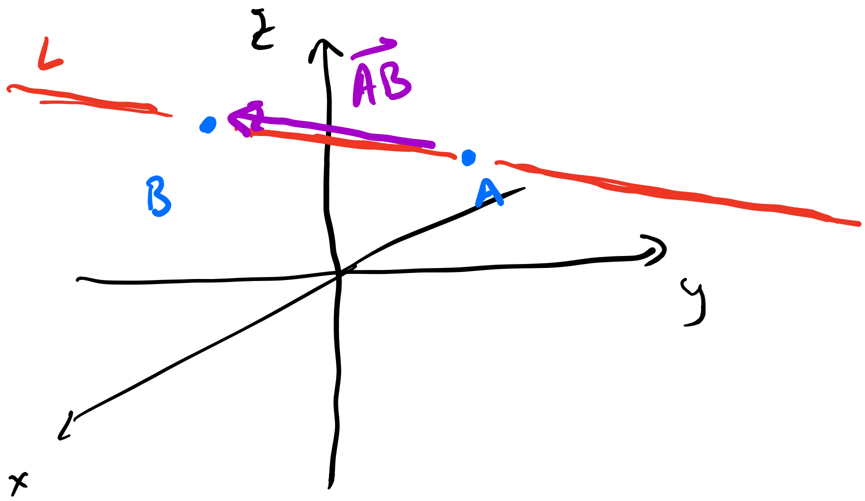
Then $(*)$ becomes:

$$\boxed{x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc.}$$

<u>parametric equations for L</u>.

<u>Ex</u>. Write equations for line L thru A, B

$$A = (1,2,3), \qquad B = (-2,-7,4).$$



point on L? $A = \langle 1,2,3\rangle$ B

vector parallel to L? $\vec{AB}$

$$\vec{AB} = \langle -2-1, -7-2, 4-3\rangle$$
$$= \langle -3, -9, 1\rangle.$$

$$\underline{r} = \underline{r}_0 + t\underline{v}$$

in our case ⇝ $\boxed{\underline{r} = \langle 1,2,3\rangle + t\langle -3,-9,1\rangle.}$ vector equation

$\langle x,y,z\rangle$

$\boxed{x = 1-3t, \quad y = 2-9t, \quad z = 3+t.}$ parametric equations.

Intersection w/ xz-plane? $(1/3, 0, 29/9)$.

3rd way to represent lines.

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc.$$

$$\Rightarrow t = \frac{x-x_0}{a} \qquad \Rightarrow t = \frac{y-y_0}{b} \qquad \Rightarrow t = \frac{z-z_0}{c}$$

$$\Rightarrow \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.}$$ symmetric equations for L.

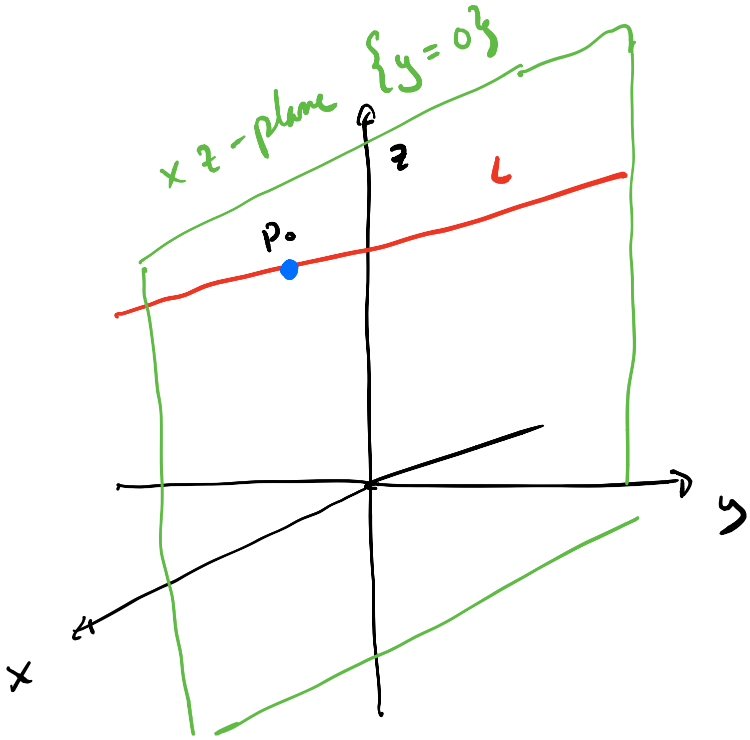
Ex. $x = 1-3t, \quad y = 2-9t, \quad z = 3+t.$

$$\Rightarrow t = -\frac{x-1}{3} \qquad t = -\frac{y-2}{9} \qquad \Rightarrow t = z-3$$

$$\Rightarrow \boxed{-\frac{x-1}{3} = -\frac{y-2}{9} = z-3}.$$

OH: 2-3.

§10.5: Equations of lines and planes.



$x = 1-3t,\ y = 2-9t,\ z = 3+t.$

Q: How to find intersection w/ xz-plane.

$t_0 :=$ t-value where L hits xz-plane.

$P_0 = (x_0, 0, z_0).$

$y_0 = 0 \Rightarrow 2 - 9t_0 = 0$

$\Rightarrow t_0 = 2/9.$

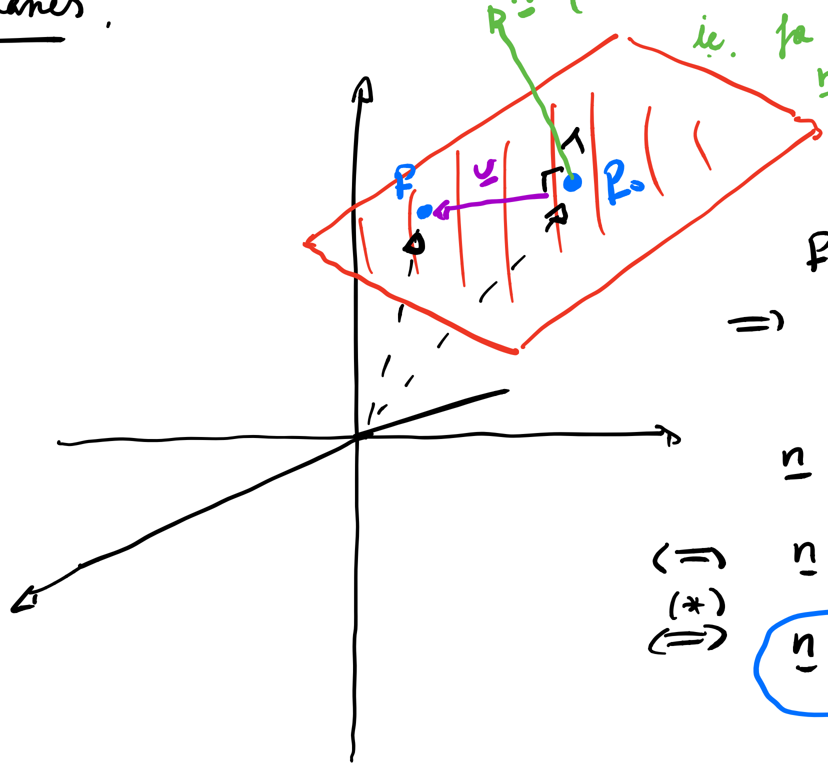
$\Rightarrow x_0 = 1 - 3t_0 = 1 - 3 \cdot 2/9 = 1/3.$

$z_0 = 3 + t_0 = 3 + 2/9 = 29/9.$

$\Rightarrow P_0 = (1/3,\ 0,\ 29/9).$

Planes.

$\underline{n}$ (normal vector to plane, ie. for all vector in plane, $\underline{n}$ is perp. to that vector)



$$\underline{p} = \underline{p}_0 + \underline{v}$$

$$\Rightarrow \underline{v} = \underline{p} - \underline{p}_0 . \quad (*)$$

$$\underline{n} \perp \underline{v}$$

$$\Leftrightarrow \underline{n} \cdot \underline{v} = 0$$

$$\overset{(*)}{\Leftrightarrow} \boxed{\underline{n} \cdot (\underline{p} - \underline{p}_0) = 0 .} \quad (**)$$

Or equivalently:

$\underline{n} = \langle a, b, c \rangle$

$\underline{p}_0 = \langle x_0, y_0, z_0 \rangle$

$\underline{p} = \langle x, y, z \rangle$.

unknowns

$(**)$ becomes $\underbrace{\langle a,b,c \rangle \cdot (\langle x,y,z \rangle - \langle x_0,y_0,z_0 \rangle)}_{=0}$

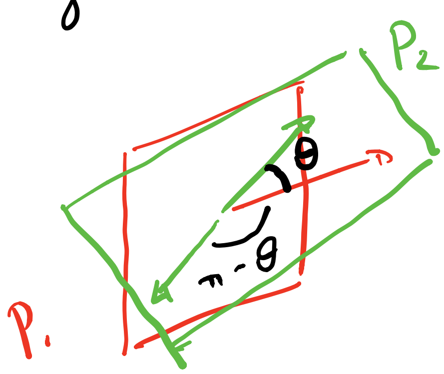
$$\Updownarrow$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 .}$$

(Simpler equation, setting $d := ax_0 + by_0 + cz_0$:

$$\boxed{ax + by + cz = d .}$$)

The angle between two planes P_1, P_2 is the acute angle between their normal lines.



use: a normal vector to $\{ax + by + cz = d\}$ is $\langle a, b, c \rangle$.

Fall '16, #1a. $P_1 := \{x + y + 0 \cdot z = 27\}$, $P_2 := \{2x + z = 10\}$.

$\underline{n}_1 = \langle 1, 1, 0 \rangle$ $\underline{n}_2 = \langle 2, 0, 1 \rangle$

$Q = (3, 4, 1)$.

Write equation for plane P_3 that's perpendicular to P_1, P_2, contains Q.

i.e. $\underline{n}_3 \perp \underline{n}_1$, $\underline{n}_3 \perp \underline{n}_2$.

need:
- point on P_3 $Q = (3,4,1)$
- vector $\underline{n}_3$ $\perp$ to P_3 $\langle 1, -1, -2 \rangle$.

Let's set $\underline{n}_3 := \underline{n}_1 \times \underline{n}_2$.

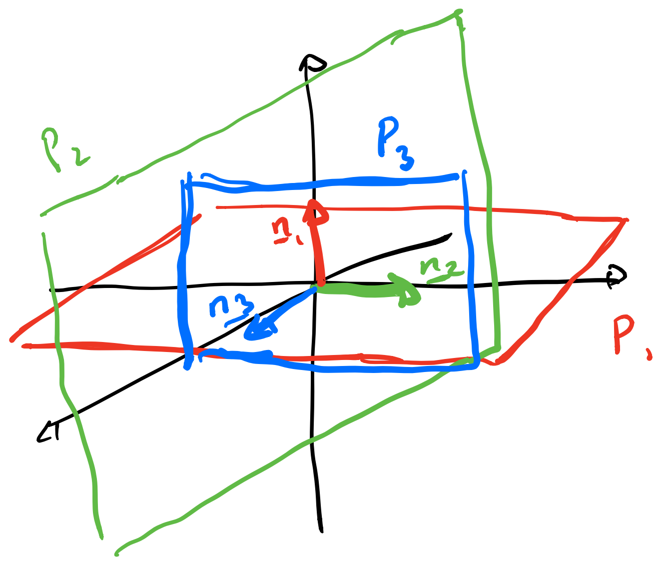
$$= \langle 1,1,0 \rangle \times \langle 2,0,1 \rangle$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \langle 1, -1, -2 \rangle$$

$\rightsquigarrow$ $1 \cdot (x-3) + (-1) \cdot (y-4) + (-2) \cdot (z-1) = 0$

$\Rightarrow$ $x - 3 - y + 4 - 2z + 2 = 0$

$\Rightarrow$ $\boxed{x - y - 2z = -3}$. $\triangle$



§10.6: Cylinders, quadric surfaces.

Goal: Given a quadric surface

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

develop technique to draw it.

Simpler but equivalent goal: Do this just for surfaces of the form:

$$Ax^2 + By^2 + Cz^2 + D = 0$$

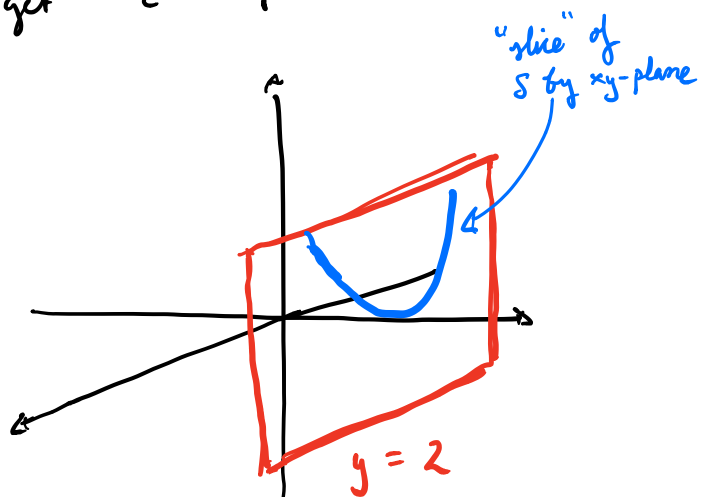
or

$$Ax^2 + By + Cz = 0.$$

Ex 1. Sketch surface "S" $z = x^2$.

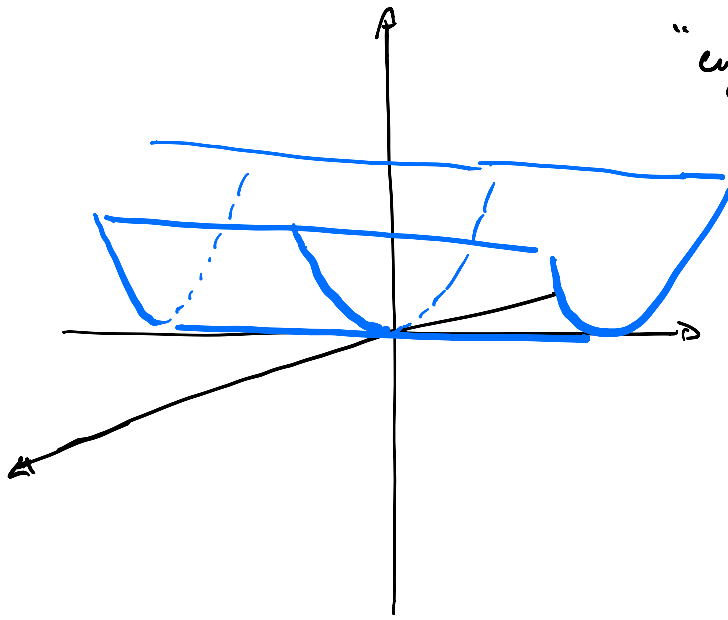
Let's use the method of traces. What's the intersection of S with the plane $\{y = c\}$ (parallel to xz-plane)?

Plug $z = c$ in to $z = x^2$; still get $z = x^2$, but now describes a curve in $\{z = c\}$.



So, each slice is the curve $z = x^2$ in $\{y = c\}$.

$\Rightarrow$ S is the result of taking a single copy of that curve and "sweeping S out" by dragging it in direction of y-axis.

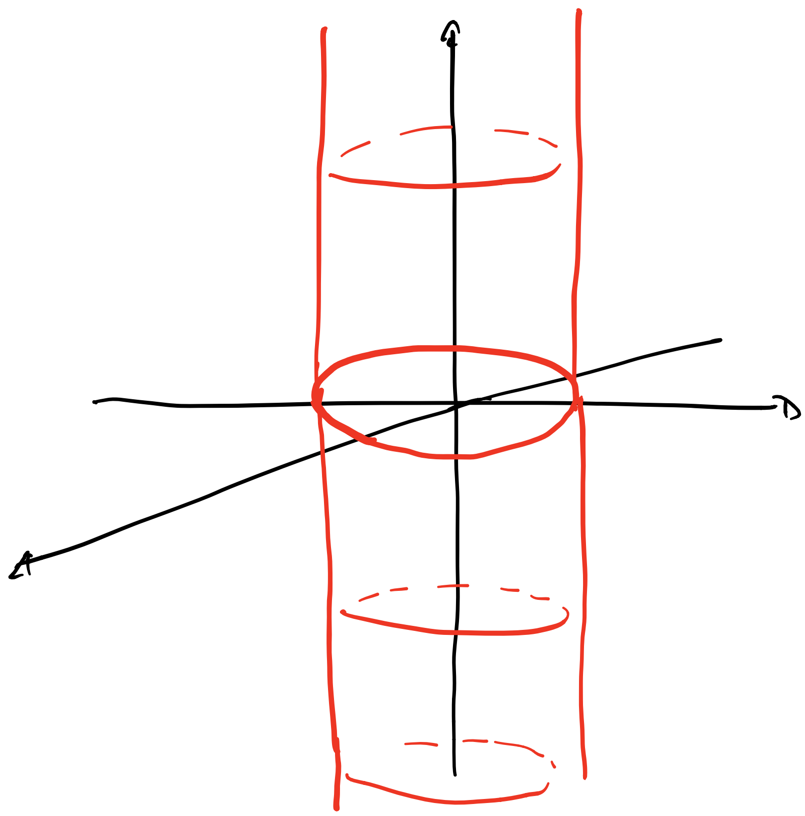


"cylinder" (in a generalized sense)

If we have a surface S defined by an equation $f(x,z) = 0$, then we can draw S by drawing the curve $f(x,z) = 0$ in the xz-plane, then "sweeping S out" by dragging the curve in the direction of the y-axis.

(similarly for a surface $f(x,y) = 0$, or $f(y,z) = 0$)

BTW, why are these called "cylinders"? Well, consider the surface defined by $x^2 + y^2 = 1$.



Drawing more general quadric surfaces. We can use the "method of traces" to draw more general quadric surfaces.

Ex. Draw $x^2+y^2+z^2=4$. $(*)$

Q. First need to choose: slice by $\{x=c\}$, $\{y=c\}$, or $\{z=c\}$?

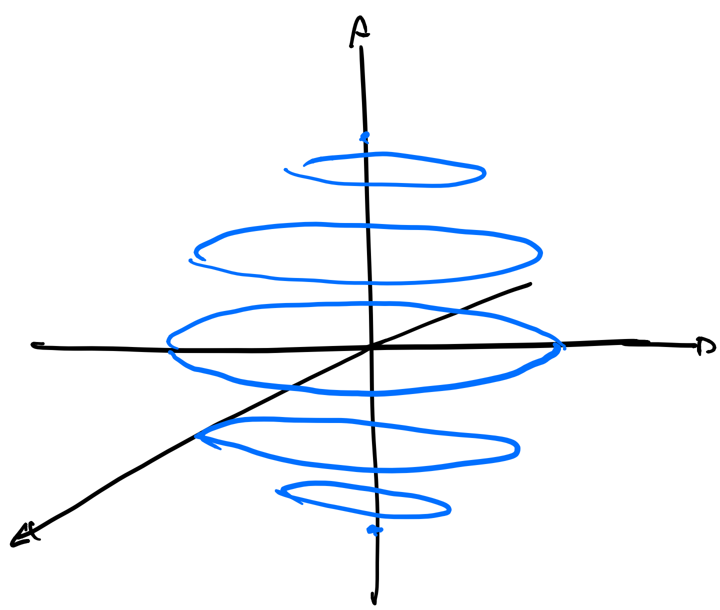
Here, doesn't matter.

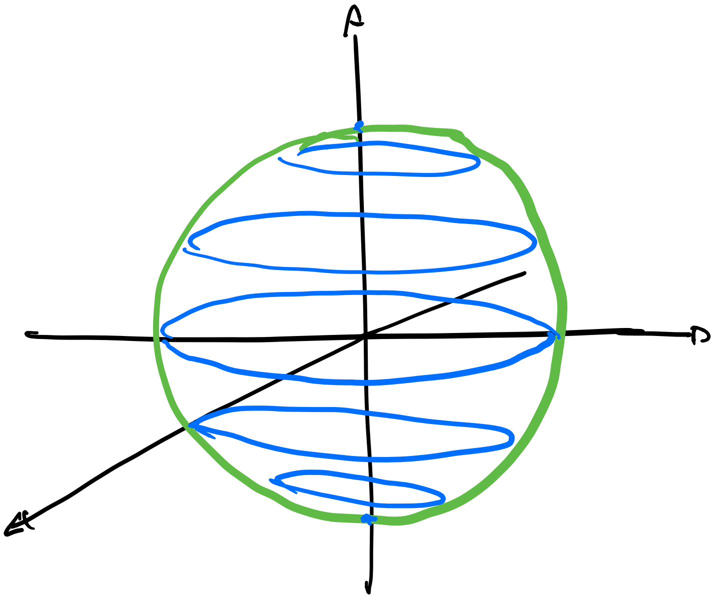
Slice by $\{x=c\}$? Plug in $x=c$ to $(*)$

$$\rightsquigarrow x^2+y^2+c^2=4$$
$$\Rightarrow x^2+y^2 = 4-c^2.$$

$$\Rightarrow x=c \text{ slice of } S \text{ is} \ldots \begin{cases} \text{radius} - \sqrt{4-c^2} \text{ circle if } & -2<c<2 \\ \text{point } \{(0,0)\} \text{ if } & c=2,-2 \\ \emptyset \text{ if } & |c|>2 \end{cases}$$

And now we can draw S!





(a sphere!)

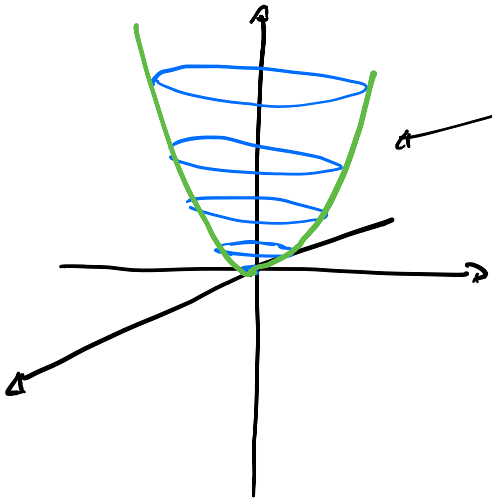
$\triangle$

Ex 4. Draw $z = 4x^2 + y^2$.

a. $\boxed{\text{Option 1: } z\text{-slices}}$.

Intersect S w/ $\{z = c\}$ $\rightsquigarrow$ curve $c = 4x^2 + y^2$ in xy-plane.

$$c = 4x^2 + y^2 \text{ is } \ldots \begin{cases} \text{an ellipse if } c > 0 \\ \{(0,0)\} \text{ if } c = 0 \\ \emptyset \text{ if } c < 0. \end{cases}$$

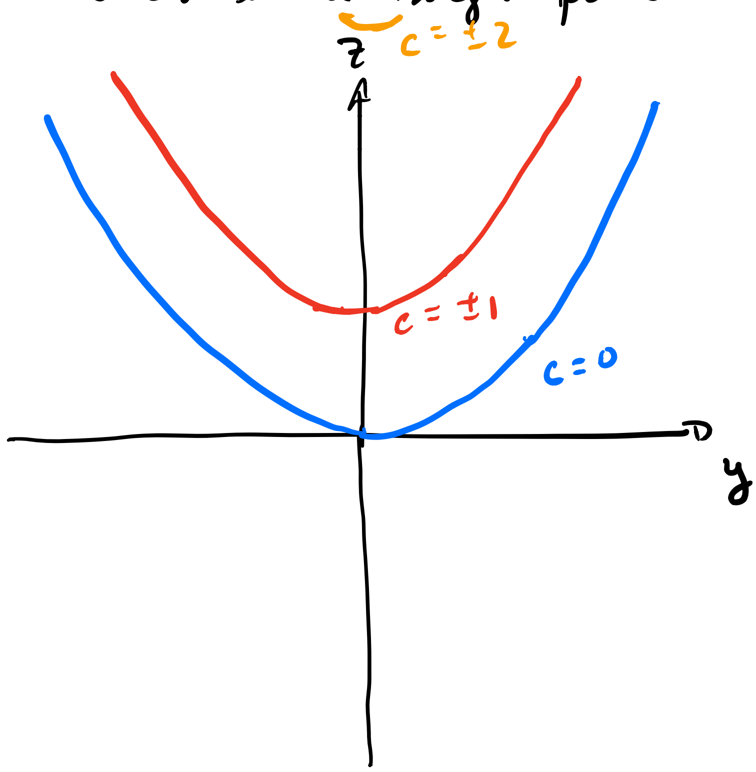


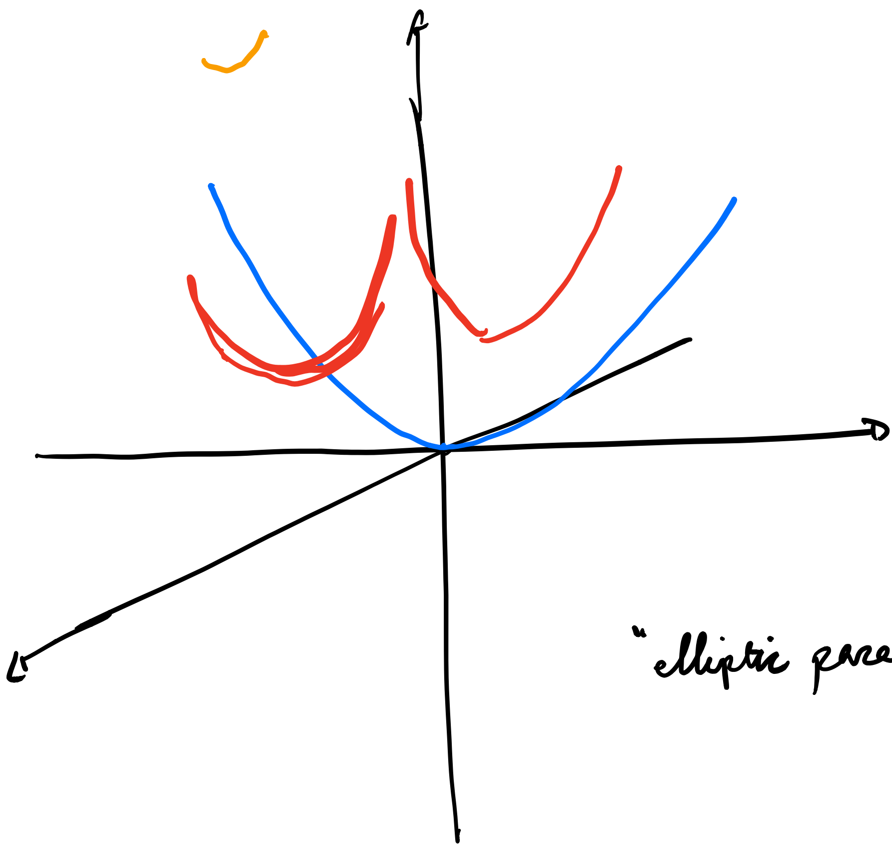
BTW, how to find the "profile"? Try a single add'l slice: $x = 0$ $\rightsquigarrow$ $z = y^2$ (parabola!).

Option 2: x-slices

$$z = 4x^2 + y^2 \xrightarrow{x=c} z = y^2 + 4c^2.$$

If we plot all traces in a single plane:





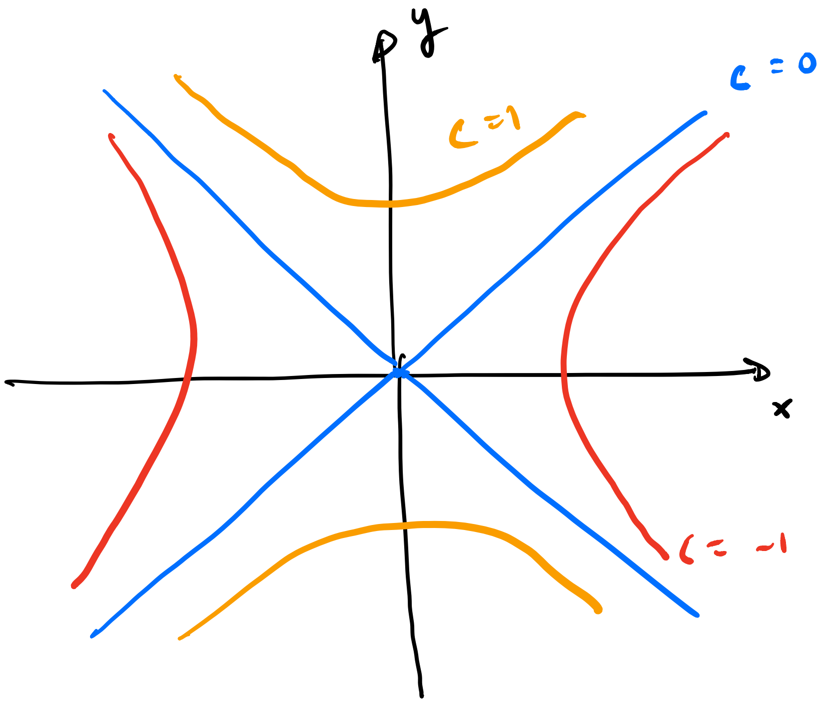
"elliptic paraboloid".

Δ

How about a trickier example...

Ex 5. $z = y^2 - x^2$.

Let's take z-slices. $z = y^2 - x^2 \xrightarrow{z=c} y^2 - x^2 = c$.



Let's look at Mathematica. Δ.