Math 225 Final

Show all work and carefully justify your answers if you wish to receive credit.

Notes and/or calculators are not permitted.

Write legibly, and make sure that you scanned solutions are easy to read before you upload them.

1) Let V be an n dimensional vector space, W an (n-1) dimensional vector space, and $T: V \to W$ a linear transformation with a one dimensional null space.

a) If $\{v_1, \dots, v_n\}$ spans V show that $\{T(v_1), \dots, T(v_n)\}$ spans W.

b) Is $\{T(v_1), \dots, T(v_n)\}$ a basis for W? Carefully explain your answer. (15 points)

2) Which of the following matrices are similar to one another? Carefully justify your answer.

a) $\begin{vmatrix} 9 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{vmatrix}$ b) $\begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 1 & 9 \end{vmatrix}$ c) $\begin{vmatrix} 7 & 6 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ d) $\begin{vmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 2 & 7 \end{vmatrix}$ e) $\begin{vmatrix} 3 & 0 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 3 \end{vmatrix}$. (15 points)

3) Let $A = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix}$. (15 points)

a) The vector $\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$ is an eigenvector for A corresponding to the eigenvalue 2. Find a generalized eigenvector.

b) What is the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$?

4) Let $A = \begin{vmatrix} 5 & 1 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$. (20 points)

a) What are the eigenvalues of A? For each eigenvalue λ find a basis for the λ-eigenspace.

b) Is A diagonalizable? Why?

If your answer is yes find a matrix that diagonalizes A.

5) a) Find the general solution to $(D^2+4)(D-3)^2y = 0$.

b) Find the general solution to $(D^2+4)(D-3)^2y = e^{2x}$. (20 points)

6) Let T: $\mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying T(1,1) = (2,5), and T(3,1) = (2,1). What is T(5,3)? (15 points)

7) Let $P_2(x)$ be the vector space of polynomials of degree ≤ 2, and let

$B = \{1-x, 1+x, 1+x^2\}$, $C = \{2, x, x^2\}$ Let T: $P_2(x) \to P_2(x)$ be the linear transformation

$T(ax^2+bx+c) = (a+b)x^2 + cx + b$.

a) Find the matrix of T with respect to B.

b) Find the change of basis matrix if we change from the basis B to the basis C. (15 points)

8) The transformation T: $R^3 \to R^3$ has the matrix

$$A = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 6 & 4 & 9 \end{vmatrix} .$$

(20 points)

a) Without computing A^{-1} explain why T is invertible.

b) What is the rank of T? Why?

c) What is the nullity of T? Why?

d) Using Gaussian elimination compute the matrix of T^{-1}.

(9) Let $A = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$, and let $\mathbf{b} = \begin{vmatrix} e^t \\ e^{-t} \end{vmatrix}$. (15 points)

Find a particular solution to $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}$.