Quiz 1

Math 225

8/27/20

Problem 1. (5 pts) Show by example that the product of two symmetric matrices need not be symmetric.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Problem 2.

(5 pts) Use elementary row operations to put the matrix $A = \begin{bmatrix} -2 & -6 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 6 \\ 1 & 3 & 0 & -1 \end{bmatrix}$ into row-echelon form.

$$\begin{bmatrix} -2 & -6 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 6 \\ 1 & 3 & 0 & -1 \end{bmatrix} \underset{\sim}{P_{14}} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 6 \\ -2 & -6 & -1 & 3 \end{bmatrix} \underset{\sim}{A_{14}(2)} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -1 & 1 \end{bmatrix} \underset{\sim}{P_{23}} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\underset{\sim}{M_2(1/3)} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \underset{\sim}{A_{24}(1)} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \underset{\sim}{A_{34}(-3)} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3.

(5 pts) Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 0 & -1 & -1 \end{bmatrix}$ and show that $A^3 - 3A + 2I = 0$ where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix.

$$A^3 = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -6 & 9 \\ 0 & -2 & -6 \\ 0 & -3 & -5 \end{bmatrix}$$

So

$$A^3 - 3A + 2I = \begin{bmatrix} 1 & -6 & 9 \\ 0 & -2 & -6 \\ 0 & -3 & -5 \end{bmatrix} + \begin{bmatrix} -3 & 6 & -9 \\ 0 & 0 & 6 \\ 0 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Quiz 2

Math 225

9/3/20

Problem 1. (4 pts) Find the determinant of $A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$.

$$\begin{aligned} \det(A) &= 3\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix} \\ &= 3(-(-1)) + 2(-4) + 2 \\ &= 3 - 8 + 2 \\ &= -3 \end{aligned}$$

Problem 2. (5 pts) Find the inverse of $A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} -3 & -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{M_1(2), A_{21}(3)}{\sim} \left[\begin{array}{ccc|ccc} 0 & -4 & 5 & 2 & 3 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{A_{31}(-4)}{\sim} \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 2 & 3 & -4 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{M_2(3), A_{12}(1)}{\sim} \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 2 & 3 & -4 \\ -6 & 0 & 0 & 2 & 6 & -4 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{M_3(3), A_{13}(2)}{\sim} \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 2 & 3 & -4 \\ -6 & 0 & 0 & 2 & 6 & -4 \\ 0 & -3 & 0 & 4 & 6 & -5 \end{array}\right]$$

$$\overset{P_{12}, P_{23}}{\sim} \left[\begin{array}{ccc|ccc} -6 & 0 & 0 & 2 & 6 & -4 \\ 0 & -3 & 0 & 4 & 6 & -5 \\ 0 & 0 & -3 & 2 & 3 & -4 \end{array}\right]$$

$$\overset{M_1(-\frac{1}{6}), M_2(-\frac{1}{3}), M_3(-\frac{1}{3})}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -1 & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{4}{3} & -2 & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & -1 & \frac{4}{3} \end{array}\right]$$

So A^{-1} exists and

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & -1 & \frac{2}{3} \\ -\frac{4}{3} & -2 & \frac{5}{3} \\ -\frac{2}{3} & -1 & \frac{4}{3} \end{bmatrix}$$

Problem 3.
(6 pts) Find all values of a and b for which the system

$$\begin{aligned} ax_1 + 2x_2 &= 3 \\ x_1 - x_2 &= b \end{aligned}$$

i) is inconsistent

ii) has infinite solutions

iii) has one unique solution

$$\left[\begin{array}{cc|c} a & 2 & 3 \\ 1 & -1 & b \end{array}\right] \overset{A_{21}(2)}{\sim} \left[\begin{array}{cc|c} a+2 & 0 & 3+2b \\ 1 & -1 & b \end{array}\right]$$

$$\overset{M_2(-(a+2))}{\sim} \left[\begin{array}{cc|c} a+2 & 0 & 3+2b \\ -(a+2) & a+2 & -b(a+2) \end{array}\right]$$

$$\overset{A_{12}(1)}{\sim} \left[\begin{array}{cc|c} a+2 & 0 & 3+2b \\ 0 & a+2 & b(a+2)+(3+2b) \end{array}\right]$$

i) To be inconsistent, we need $\text{rank}(A) < \text{rank}(A^{\#})$. It must be that $a+2=0$ and $2b+3 \neq 0$. So the answer is $a=-2$ and $b \neq -\frac{3}{2}$

ii) To have infinite solutions we need $\text{rank}(A) = \text{rank}(A^{\#}) < 2$ which is the number of variables. It must be that $a+2=0$ and $2b+3=0$. So the answer is $a=-2$ and $b=-\frac{3}{2}$.

iii) To have one unique solution we need $\text{rank}(A) = \text{rank}(A^{\#}) = 2$. To do this, we just need $a+2 \neq 0$. Then, for any given value of b, we will have a unique solution. The answer is $a \neq -2$ and b can be any real value.

Quiz 3

Math 225

9/10/20

Problem 1. (5 pts) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} a & b-a \\ 3c & 3d-3c \end{bmatrix}$. Use properties of the determinant to find $\det(-A^{-1}B)$ without expanding out the determinants. (Hint: First find $\det(B)$ in terms of $\det(A)$.)

Problem 2. (5 pts) Use Cramer's rule to solve the system

$$\begin{aligned} -x_1 + 4x_3 &= 1 \\ 3x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 - x_3 &= -2 \end{aligned}$$

for x_1 and x_2 (you need not find x_3).

Problem 3. (5 pts) Find the adjoint of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 5 \end{bmatrix}$ with $\det(A) = -4$ and use it to find the inverse.

Quiz 4

Math 225

9/17/20

Problem 1. (4 pts) Let $S = \mathbb{R}$. Define addition in S by

$$x \oplus y = -2x - 2y$$

and scalar multiplication is the standard multiplication of real numbers. Show that S is <u>not</u> a vector space over $\mathbb{R}$. (***Hint:*** *Check the associative property of addition*)
The associative property of addition says that we need $x \oplus (y \oplus z) = (x \oplus y) \oplus z$. However,

$$\begin{aligned} x \oplus (y \oplus z) &= x \oplus (-2y - 2z) \\ &= -2x - 2(-2y - 2z) \\ &= -2x + 4y + 4z \end{aligned} \qquad \begin{aligned} (x \oplus y) \oplus z &= (-2x - 2y) \oplus z \\ &= -2(-2x - 2y) - 2z \\ &= 4x + 4y - 2z \end{aligned}$$

Since it is not always true that $x = z$, S is not a vector space with this addition operation.

Problem 2. (6 pts) Determine whether or not the following sets S are subspaces of the given vector spaces V.

i) $V = \mathbb{R}^2$, $S = \left\{ \begin{bmatrix} a+1 \\ a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$.

The zero vector in V is $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. However, this is not an element of S since there is no $a \in \mathbb{R}$ so $a + 1 = 0$ and $a = 0$. So **No!** S is not a subspace of V.

ii) $V = \mathbb{P}_2(\mathbb{R})$ the set of polynomials of degree 2 or less over $\mathbb{R}$, $S = \{a + bx^2 \mid a, b \in \mathbb{R}\}$

0) S is a subset of V.

1) Let $a + bx^2$ and $c + dx^2$ be in S. Then

$$(a + bx^2) + (c + dx^2) = (a + c) + (b + d)x^2$$

which is in S. Since addition of polynomials is associative and commutative in V, and S is a subset of V, addition in S is also associative and commutative.

2) Let $c \in \mathbb{R}$ and $a + bx^2 \in S$, then $c(a + bx^2) = ca + cbx^2$ which is in S.

So **Yes!** S is a subspace of V.

iii) $V = \mathrm{Sym}_2(\mathbb{R})$ the set of 2×2 symmetric matrices over $\mathbb{R}$, $S = \left\{ \begin{bmatrix} a & a-b \\ b-a & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

The matrix $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ where $a = 1$ and $b = 2$. However, this matrix is not symmetric, so S is not a subset of V, thus **No!** S is not a subspace of V.

Problem 3. (5 pts) Find the null space Null(A) and then a spanning set S for Null(A) where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

The null space is the set of all vectors so $A\vec{x} = \vec{0}$. Thus, we will put A into RREF.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \overset{A_{12}(-5)}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{bmatrix} \overset{M_2(-\frac{1}{4})}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \overset{A_{21}(-2)}{\sim} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Thus, we get the system

$$\begin{aligned} x_1 - x_3 - 2x_4 &= 0 \\ x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

So

$$\text{Null}(A) = \left\{ \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_3, x_4 \in \mathbb{R} \right\}$$

Now, we can write

$$\begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_4 \\ -3x_4 \\ 0 \\ x_4 \end{bmatrix}$$

and so a spanning set is

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Quiz 5

Math 225

9/24/20

Problem 1. (5 pts) Check whether the set

$$S = \{(2,2,0,0),(3,1,-1/6,2/3),(0,3,1/4,-1)\}$$

is linearly independent or linearly dependent. Can there exist a basis for $\mathbb{R}^4$ that contains this set?

We must find the number of solutions to

$$\begin{bmatrix} 2 & 3 & 0 \\ 2 & 1 & 3 \\ 0 & -1/6 & 1/4 \\ 0 & 2/3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ -1/6 \\ 2/3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \\ 1/4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We have

$$\begin{bmatrix} 2 & 3 & 0 \\ 2 & 1 & 3 \\ 0 & -1/6 & 1/4 \\ 0 & 2/3 & -1 \end{bmatrix} \overset{A_{12}(-1)}{\sim} \begin{bmatrix} 2 & 3 & 0 \\ 0 & -2 & 3 \\ 0 & -1/6 & 1/4 \\ 0 & 2/3 & -1 \end{bmatrix} \overset{\substack{M_1(1/2) \\ M_2(-1/2)}}{\sim} \begin{bmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -3/2 \\ 0 & -1/6 & 1/4 \\ 0 & 2/3 & -1 \end{bmatrix} \overset{\substack{A_{23}(1/6) \\ A_{24}(-2/3)}}{\sim} \begin{bmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the existence of a free variable indicating that there are multiple solutions and so S is linearly dependent. Any set containing S will also be linearly dependent and therefore cannot be a basis.

Problem 2. (5 pts) Use the Wronskian to show that the functions $f_1(x) = \sin(x)$, $f_2(x) = e^x$, $f_3(x) = e^{-x}$ are linearly independent on the interval $I = (-\pi, \pi)$.

$$W[f_1,f_2,f_3](x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} \sin(x) & e^x & e^{-x} \\ \cos(x) & e^x & -e^{-x} \\ -\sin(x) & e^x & e^{-x} \end{vmatrix}$$

(One might further simplify the Wronskian to have an easier time.) Take $x = \frac{\pi}{2}$ to get

$$W[f_1,f_2,f_3](\pi/2) = \begin{vmatrix} 1 & e^{\pi/2} & e^{-\pi/2} \\ 0 & e^{\pi/2} & -e^{-\pi/2} \\ -1 & e^{\pi/2} & e^{-\pi/2} \end{vmatrix} = \begin{vmatrix} e^{\pi/2} & -e^{-\pi/2} \\ e^{\pi/2} & e^{-\pi/2} \end{vmatrix} - \begin{vmatrix} e^{\pi/2} & e^{-\pi/2} \\ e^{\pi/2} & -e^{-\pi/2} \end{vmatrix} = 2-(-2) = 4 \neq 0$$

Since the Wronskian takes a nonzero value on the interval, the functions are linearly independent.

Problem 3. (5 pts) Find a basis and the dimension for the subspace

$$S = \{p(x) \in P_3(\mathbb{R}) : p(1) = 0\}$$

of $P_3(\mathbb{R})$ the polynomials of degree at most 3. (Hint: How can you rewrite the condition $p(1) = 0$?)

For a polynomial $p(x) = ax^3 + bx^2 + cx + d \in P_3(\mathbb{R})$, the condition $p(1) = 0$ is equivalent to $a + b + c + d = 0$. So

$$\begin{aligned} S &= \{ax^3 + bx^2 + cx + d \in P_3(\mathbb{R}) : a + b + c + d = 0\} \\ &= \{(-b - c - d)x^3 + bx^2 + cx + d : b, c, d \in \mathbb{R}\} \\ &= \{b(-x^3 + x^2) + c(-x^3 + x) + d(-x^3 + 1) : b, c, d \in \mathbb{R}\} \end{aligned}$$

Thus, $B = \{-x^3 + x^2, -x^3 + x, -x^3 + 1\}$ spans S, and one can see they are linearly independent because in order to have a linear combination

$$b(-x^3 + x^2) + c(-x^3 + x) + d(-x^3 + 1) = (-b - c - d)x^3 + bx^2 + cx + d$$

be zero, we see that the coefficients of $x^2, x, 1$ which are b, c, d must all be zero. So B is a basis for S and $\dim(S) = |B| = 3$.

Quiz 6

Math 225

10/01/20

Problem 1. (5 pts) Let

$$B = \left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \end{bmatrix} \right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$$

be bases for $\mathbb{R}^2$.

(i) Find the change of basis matrix from B to C, $P_{C \leftarrow B}$.

Say $B = \{v_1, v_2\}$ and $C = \{w_1, w_2\}$. Then $v_1 = w_1 + w_2$ and $v_2 = w_1 - w_2$ so

$$[v_1]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [v_2]_C = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \Longrightarrow \qquad P_{C \leftarrow B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(ii) Find $\left[\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right]_B$ and use your answer in (i) to find $\left[\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right]_C$.

$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2v_1 + v_2$ so $\left[\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Finally,

$$\left[\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right]_C = P_{C \leftarrow B} \left[\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right]_B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Problem 2. (4 pts) Let V be a 2-dimensional real vector space over $\mathbb{R}$ with three bases, A, B, C. Given that

$$P_{C \leftarrow A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad P_{B \leftarrow A} = \begin{bmatrix} -1 & 0 \\ -2 & 2 \end{bmatrix}$$

(i) Find $(P_{B \leftarrow A})^{-1}$

$$(P_{B \leftarrow A})^{-1} = \begin{bmatrix} -1 & 0 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$$

(ii) Find $P_{C \leftarrow B}$ ***Hint:*** $P_{C \leftarrow A} = P_{C \leftarrow B} P_{B \leftarrow A}$.

Using the hint, $P_{C \leftarrow B} = P_{C \leftarrow A}(P_{B \leftarrow A})^{-1}$, so from (i) we have

$$P_{C \leftarrow B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -7 & 2 \end{bmatrix}$$

Problem 3. (6 pts) Find a basis for the rowspace and colspace of A for

$$A = \begin{bmatrix} -2 & 4 & 10 & 5 & -3 \\ 1 & -2 & -5 & 2 & 3 \\ 7 & -14 & -35 & 6 & 5 \\ -1 & 2 & 5 & 7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & 10 & 5 & -3 \\ 1 & -2 & -5 & 2 & 3 \\ 7 & -14 & -35 & 6 & 5 \\ -1 & 2 & 5 & 7 & 0 \end{bmatrix} \overset{A_{21}(2), A_{23}(-7), A_{24}(1)}{\sim} \begin{bmatrix} 0 & 0 & 0 & 9 & 3 \\ 1 & -2 & -5 & 2 & 3 \\ 0 & 0 & 0 & -8 & -16 \\ 0 & 0 & 0 & 9 & 3 \end{bmatrix}$$

$$\overset{A_{14}(-1), M_1(\frac{1}{3}), M_3(-\frac{1}{8})}{\sim} \begin{bmatrix} 0 & 0 & 0 & 3 & 1 \\ 1 & -2 & -5 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{A_{31}(-3), M_1(-\frac{1}{5})}{\sim} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & -5 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{P_{13}, P_{12}}{\sim} \begin{bmatrix} 1 & -2 & -5 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, we get that that the row space is spanned by all the nonzero rows of A in REF

$$\text{rowspace}(A) = \text{Span} \underbrace{\left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{basis for the row space}} \subseteq \mathbb{R}^5$$

and the column space is spanned by the columns of A associated to the pivot columns of A in REF

$$\text{colspace}(A) = \text{Span} \underbrace{\left\{ \begin{bmatrix} -2 \\ 1 \\ 7 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ 0 \end{bmatrix} \right\}}_{\text{basis for the column space}} \subseteq \mathbb{R}^4$$

Quiz 7

Math 225

10/15/20

Problem 1. (5 pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation satisfying $T(-3, 1) = (2, 0, 5)$ and $T(0, -2) = (0, 6, 2)$. Find the matrix of T and the kernel of T.

We have

$$\begin{aligned} T((1,0)) &= T(-\frac{1}{3}(-3,1) - \frac{1}{6}(0,-2)) \\ &= -\frac{1}{3}T((-3,1)) - \frac{1}{6}T((0,-2)) \\ &= -\frac{1}{3}(2,0,5) - \frac{1}{6}(0,6,2) \\ &= (-\frac{2}{3}, -1, -2) \end{aligned}$$

and

$$T((0,1)) = T(-\frac{1}{2}(0,-2)) = -\frac{1}{2}T((0,-2)) = -\frac{1}{2}(0,6,2) = (0,-3,-1)$$

So the matrix of T is $\begin{bmatrix} -2/3 & 0 \\ -1 & -3 \\ -2 & -1 \end{bmatrix}$. The kernel of T is then the null space of the matrix which is easily seen to be $\{(0,0)\}$ by looking at the corresponding homogenous system of equations.

Problem 2. (5 pts) Let A be a 4×3 matrix such that $A \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. Show that the dimension of the row space of A is at most 2.

We have $A \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} = A \left(\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right) = \vec{0}$ so $\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} \in \ker(A)$ implying that $\text{nullity}(A) \geq 1$.
Then by the rank-nullity theorem, $\text{rank}(A) = 3 - \text{nullity}(A) \leq 2$. Since the dimension of the row space of A is $\text{rank}(A)$ we are done.

Problem 3. (5 pts) Let $T : M_2(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a-d \\ b+c \end{bmatrix}$. Is T injective? Surjective? An isomorphism? Why or why not?

Take, for instance, $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$ to see that T is not injective, hence not an isomorphism.
Now for any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ see that $T\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}$ so that $\begin{bmatrix} x \\ y \end{bmatrix}$ is in the range of T. This shows T is surjective.

Quiz 8

Math 225

10/22/20

Problem 1. (7 pts)
Let

$$T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^3$$
$$p(x) \mapsto \begin{bmatrix} p(0) \\ p(1) \\ p(-1) \end{bmatrix} \qquad \left(\text{For example, } T(2x+4) = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} \right).$$

(a) Given basis $B = \{1+x^2, 1-x^2, 1+x+x^2\}$ for $\mathbb{P}_2(\mathbb{R})$ and basis $C = \left\{ \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$, find $[T]_B^C$

Let $C = \{v_1, v_2, v_3\}$. By inspection,

$$[T(1+x^2)]_C = \left[\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right]_C = -v_1$$

$$[T(1-x^2)]_C = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]_C = -v_1 - 2v_2$$

$$[T(1+x+x^2)]_C = \left[\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right]_C = -v_1 + v_2 + 2v_3$$

So

$$[T]_B^C = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Find the eigenvalues of $A = [T]_B^C$. Is A defective or nondefective?

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 & -1 \\ 0 & -2-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda)(2-\lambda) = -(\lambda+1)(\lambda+2)(\lambda-2)$$

So the eigenvalues of A are $-1, -2, 2$. Since A is 3×3 and has 3 distinct eigenvalues, A is nondefective.

Problem 2. (4 pts) Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

Given that A has characteristic polynomial $p(\lambda) = \lambda^2(\lambda - 3)(\lambda - 1)$ determine if A is defective or nondefective.

A has 3 distinct eigenvalues. So A will be nondefective if and only if $\lambda = 0$ as a 2-dimensional associated eignespace.

Since $A - 0I = A$, this amounts to finding the nullspace of A.

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \overset{A_{12}(-1), A_{13}(-1)}{\sim} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 2 & -2 & -3 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\overset{A_{42}(2), A_{43}(1)}{\sim} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -3 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we get $x_1 = -2x_3$, $x_2 = x_3$, x_3 is free, and $x_4 = 0$. So

$$\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since this is 1-dimensional, A must be defective..

Problem 3. (4 pts) Prove that if A is invertible and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Let v be a nonzero vector in the eigenspace of A associated to λ. Then

$$\begin{aligned} Av &= \lambda v \\ A^{-1}Av &= A^{-1}\lambda v \\ v &= \lambda A^{-1} v \\ \frac{1}{\lambda} v &= A^{-1} v \qquad (*) \end{aligned}$$

Thus, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}, and v is also an eigenvector of A^{-1} associated to $\frac{1}{\lambda}$.

(*)Note that because A is invertible, it cannot have any zero eigenvalues and so $\lambda \neq 0$.

Quiz 9

Math 225

10/29/20

Problem 1. (5 pts) Find a matrix S and a diagonal matrix D such that

$$SDS^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -8 & 5 \end{bmatrix}.$$

You may use that the characteristic polynomial of the matrix is $(\lambda - 3)^2(\lambda - 1)$ and you do not need to find S^{-1}.

First we solve for the $\lambda = 3$ eigenspace:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & -8 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the $\lambda = 3$ eigenspace is $\{ \begin{bmatrix} x_1 \\ x_3/4 \\ x_3 \end{bmatrix} : x_1, x_3 \in \mathbb{R}\}$ which has basis $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/4 \\ 1 \end{bmatrix} \}$. For $\lambda = 1$ we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

so the eigenspace is $\{ \begin{bmatrix} 0 \\ x_3/2 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R}\}$ with basis $\{ \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} \}$. Thus, if $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we will have the desired relation.

Problem 2. (5 pts) Check which of the following functions $\{1, x, \sin^2(x), \cos^2(x)\}$ are contained in the kernel of $L = (\cos(x)\sin(x))D^2 + (\sin^2(x) - \cos^2(x))D$. Deduce a basis for $\ker L$.

We have $D(1) = D^2(1) = 0$ which immediately gives that $L(1) = 0$ so that $1 \in \ker L$.
Next see that $D(x) = 1$ and $D^2(x) = 0$ so

$$L(x) = \cos(x)\sin(x) \neq 0$$

so $x \notin \ker L$.
We have $D(\sin^2(x)) = 2\sin(x)\cos(x)$ and $D^2(\sin^2(x)) = 2\cos^2(x) - 2\sin^2(x)$ so

$$L(\sin^2(x)) = 2\cos(x)\sin(x)(\cos^2(x) - \sin^2(x)) + 2(\sin^2(x) - \cos^2(x))\sin(x)\cos(x) = 0$$

so $\sin^2(x) \in \ker L$.
And finally $D(\cos^2(x)) = -2\sin(x)\cos(x)$ and $D^2(\sin^2(x)) = 2\sin^2(x) - 2\cos^2(x)$ which yields

$$L(\cos^2(x)) = 2\cos(x)\sin(x)(\sin^2(x) - \cos^2(x)) - 2(\sin^2(x) - \cos^2(x))\sin(x)\cos(x) = 0$$

so $\cos^2(x) \in \ker L$. (Alternatively, noting that $\cos^2(x) = 1 - \sin^2(x)$ provides a shortcut.)
The order of the highest differential operator is 2 which means that the solution space is two dimensional. So take two linearly independent functions in the kernel, such as $\{1, \sin^2(x)\}$ to get a basis for $\ker L$. (The ratio of 1 and $\sin^2(x)$ is $\sin^2(x)$ which is not a constant.)

Problem 3. (5 pts) Determine the general solution to the differential equation $D^4 - 16 = 0$.
The auxiliary polynomial $x^4 - 16$ is a difference of squares that can be factored as $(x^2-4)(x^2+4)$ and further as $(x-2)(x+2)(x^2+4)$.
The operator $D - 2$ has $\{e^{2x}\}$ as a basis for its kernel, and $D + 2$ has $\{e^{-2x}\}$ as a basis for its kernel. On the other hand, $x^2 + 4$ has roots $\pm 2i$ so that $D^2 + 4$ has the basis $\{\sin(2x), \cos(2x)\}$ for its kernel.
Taking the span of all basis elements, we get that the general solution for $D^4 - 16 = 0$ is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \sin(2x) + c_4 \cos(2x).$$

Quiz 10

Math 225

11/05/20

Problem 1. (6 pts)

(a) Find an annihilator of e^{5x}

$A(D) = D - 5$

(b) Use the annihilator from (a) to write down a general solution to

$$(D^2 - 25)y = 40e^{5x}$$

The annihilator gives us that solutions are in $A(D)Ly = 0$ where

$$A(D)L = (D-5)(D-5)(D+5) = (D-5)^2(D+5)$$

So general solutions are of the form

$$y = ae^{5x} + bxe^{5x} + ce^{-5x} \qquad a, b, c \in \mathbb{R}.$$

Since the kernel of L has basis $\{e^{5x}, e^{-5x}\}$, we get that particular solutions come from $y_0 = bxe^{5x}$. We must find b and since

$$\begin{aligned} y_0 &= bxe^{5x} \\ y_0' &= be^{5x} + 5bxe^{5x} \\ y_0'' &= 5be^{5x} + 5be^{5x} + 25bxe^{5x} \\ &= 10be^{5x} + 25bxe^{5x} \end{aligned}$$

we get that

$$Ly_0 = 10be^{5x} + 25bxe^{5x} - 25be^{5x} = 10be^{5x} = 40e^{5x}$$

and so $b = \frac{40}{10} = 4$.

Finally, general solutions are of the form

$$y = ae^{5x} + ce^{-5x} + 4xe^{5x}.$$

Problem 2. (4 pts) Given that

$$y = ae^x \cos(2x) + be^x \sin(2x) - 2xe^x \cos(2x)$$

$$y' = (-2a + b)e^x \sin(2x) + (a + 2b - 2)e^x \cos(2x) + 4xe^x \sin(2x) - 2xe^x \cos(2x)$$

is a general solution of

$$y'' - 2y' + 5y = 8e^x \sin(2x)$$

find the solution satisfying initial conditions $y(0) = 1$ and $y'(0) = 1$.
$y(0) = a = 1$ so $a = 1$. $y'(0) = a + 2b - 2 = 1$ since $a = 1$ we get that $2b = 2$ and so $b = 1$. Thus, the solution satisfying the initial conditions is

$$y = e^x \cos(2x) + e^x \sin(2x) - 2xe^x \cos(2x).$$

Problem 3. (5 pts) Using variation of parameters, find a general solution to

$$y'' - 2y' + y = 4e^x x^{-1} \ln x \qquad x > 0.$$

(**Hints:** (1) note the derivative of $x \ln x - 1$ is $\ln x$ and (2) use the substitution $u = \ln x$ on integrals of the form $\int \frac{\ln x}{x} dx$).
since $L = (D - 1)^2$ we get that

$$\ker(L) = \text{Span}\{e^x, xe^x\}$$

which has Wronskian

$$W[y_1, y_2](x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

and so using the formula for variation of paramters, we get

$$\begin{aligned} u_1(x) &= \int \frac{xe^x 4e^x x^{-1} \ln x}{e^{2x}} dx \\ &= \int \frac{4e^{2x} \ln x}{e^{2x}} dx \\ &= \int 4 \ln x dx \\ &= 4(x \ln x - 1) \end{aligned}$$

where the last line comes from the hint. And

$$\begin{aligned} u_2(x) &= \int \frac{e^x 4e^x x^{-1} \ln x}{e^{2x}} dx \\ &= \int \frac{4 \ln x}{x} dx \\ &= \int 4u du \qquad u = \ln x \quad du = \frac{1}{x} dx \\ &= 2u^2 \\ &= 2(\ln x)^2 \end{aligned}$$

and so we get that a particular solution is $u_1 y_1 + u_2 y_2$ and so a general solution is

$$y = ae^x + bxe^x + 4e^x(x \ln x - 1) + 2xe^x(\ln x)^2.$$